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Output Feedback Controllers For Continuous-Time Switched Affine Systems

Controladores via Realimentação de Saída para Sistemas Afins com Comutação a Tempo Contínuo

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Monograph presented to the School of Mechanical Engineering of the University of Campinas in partial fulfillment of the requirements for the degree of Bachelor of Control and Automation Engineering, in the field of Control Theory.

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Abstract

Currently, there is an increasing interest within the scientific community with regard to switched dynamical systems. These systems are comprised of a finite number of subsystems and a switching function used for selecting one subsystem at each instant of time. In our context, this function is a control variable to be designed, so as to guarantee stability and a suitable performance of the overall system. In the literature, there are important contributions on this topic, however, due to the theoretical characteristics and the practical interest of this wide class of systems, there still exist several problems of great importance to be studied. In this context, our focus is to treat the problem of dynamical output feedback control design for continuous-time switched affine systems. These systems are more general when compared to switched linear systems, since they possess several equilibrium points, thus making the control design problem more challenging. As it will become clear throughout this work, our main contribution is to propose a technique for the simultaneous design of two control structures, while the existing literature considers exclusively the synthesis of the switching function. More specifically, a full-order dynamical controller and a switching function, dependent only on the measured output signal, are considered so as to guarantee stability as well as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices. The design conditions are expressed in terms of linear matrix inequalities, and do not require any stability property of each isolated subsystem. Furthermore, the proposed methodology allows for a wider range of equilibrium points to be attained, when compared to available techniques. Several examples illustrate the theory and make clear the importance of the joint action of both control structures.

Resumo

Atualmente, é crescente o interesse da comunidade científica no estudo de sistemas dinâmicos com comutação. Estes sistemas são compostos por um número finito de subsistemas e uma função de comutação que seleciona a cada instante de tempo um deles. No nosso contexto, esta função de comutação é uma variável de controle a ser projetada, de forma a garantir estabilidade e um desempenho adequado do sistema global. Na literatura existem contribuições importantes sobre este tema, entretanto, devido às suas características teóricas e o interesse prático desta abrangente classe de sistemas, ainda existem vários problemas de grande importância a serem estudados. Neste contexto, nosso interesse é tratar o projeto de controle via realimentação dinâmica de saída de sistemas afins com comutação a tempo contínuo. Estes sistemas são mais gerais quando comparados aos lineares, pois possuem vários pontos de equilíbrio formando uma região no espaço de estado, tornando assim o problema de projeto de controle mais desafiador. Como ficará claro ao longo do texto, nossa principal contribuição é propor uma técnica para o projeto simultâneo de duas estruturas de controle, enquanto que a literatura disponível considera exclusivamente a síntese da regra de comutação. Mais especificamente, um controlador dinâmico de ordem completa, e uma função de comutação, dependentes somente da saída medida, são considerados de forma a assegurar estabilidade e desempenho $\mathcal{H}_2$ e $\mathcal{H}_\infty$ do sistema global. As condições de projeto são expressas em termos de desigualdades matriciais lineares e não exigem nenhuma propriedade de estabilidade de cada subsistema isolado. Além disso a metodologia proposta permite considerar um conjunto maior de pontos de equilíbrio quando comparada às técnicas disponíveis. Vários exemplos ilustram a teoria e deixam claro a importância da ação conjunta de ambas as estruturas de controle.

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Nomenclature

Abbreviations

DC	Direct Current
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant

Latin Letters

$\mathbb{R}$	Set of real numbers.
$\mathcal{F}_\sigma$	Dynamical Filter under switching rule σ.
$\mathbb{C}$	Set of complex Numbers
$\mathbb{R}^{n\times m}$	Set of real matrices of dimension $n \times m$.
$\mathbb{K}$	Set composed of the first N positive natural numbers $\mathbb{K} := \{1, \cdot, N\}$.
$\mathcal{H}$	Set of Hurwitz matrices.
$\mathcal{L}_2$	Set of square-integrable trajectories.
$\mathbf{I}$	Identity matrix.
$v(\cdot)$	Lyapunov function.
$\mathcal{C}_\sigma$	Dynamical controller under switching rule σ.

Greek Letters

ω	Angular frequency.
σ	Switching function
Λ_N	Unit simplex of order N, $\Lambda_N := \left\{\lambda \in \mathbb{R}^N : \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1\right\}$.
λ	Arbitrary vector belonging to Λ_N.

Supercripts

$\mathbf{X}^*$	Conjugate transpose of matrix $\mathbf{X}$.
$\mathbf{X}^T$	Transpose of matrix $\mathbf{X}$.

Subscripts

$\mathbf{x}_i$	$i-$th element of vector $\mathbf{x}$.
$\mathbf{z}_{e_i}$	$i-$th element of vector $\mathbf{z}_e$.
$\mathbf{X}_i$	$i-$th matrix of the set $\{\mathbf{X}_1, \dots, N\}$.
π_{ij}	Element at row i, column j of matrix $\mathbf{\Pi}$.
$\mathbf{H}_{wz}(s)$	Transfer matrix from input $\mathbf{w}$ to output $\mathbf{z}$.
$\boldsymbol{\lambda}_0$	A given constant vector belonging to Λ_N.
$\mathbf{X}_{\lambda_0}$	Convex combination of matrices $\{\mathbf{X}_1, \dots, \mathbf{X}_N\}$ in $\boldsymbol{\lambda}_0$, $\mathbf{X}_{\lambda_0} = \sum_{i=1}^{N} \lambda_{0_i} \mathbf{X}_i$

Symbols

$\bullet$	Symmetric block of a symmetric matrix.
$\text{He}\{\mathbf{X}\}$	Hermitian operator $\text{He}\{\mathbf{X}\} := \mathbf{X} + \mathbf{X}^T$.
$\text{tr}(\mathbf{X})$	Trace of matrix $\mathbf{X}$.
$\text{diag}(\mathbf{X}, \mathbf{Y})$	Block diagonal matrix, whose elements are $\mathbf{X}$ and $\mathbf{Y}$.
$\lVert \cdot \rVert_2$	Euclidean norm.
$\lvert \cdot \rvert$	Absolute value of a scalar.
$\mathbf{X} \succ 0$ $(\mathbf{X} \prec 0)$	Matrix $\mathbf{X}$ is symmetric and positive (negative) definite, such that $\forall \mathbf{v} \neq \mathbf{0},\ \mathbf{v}^T \mathbf{X} \mathbf{v} > (<) 0$.
$\mathbf{X} \succeq 0$ $(\mathbf{X} \preceq 0)$	Matrix $\mathbf{X}$ is symmetric and positive (negative) semi-definite, such that $\forall \mathbf{v} \neq \mathbf{0},\ \mathbf{v}^T \mathbf{X} \mathbf{v} \geq (\leq) 0$.
$\min(\cdot)$	Minimum optimization problem subject to LMI restrictions.
$\sup(\cdot)$	Supremum of a set.
$\arg\ \min_{i \in \mathbb{K}}(\cdot)$	$i-$th element in $\mathbb{K}$ whose value $(\cdot)$ is minimum.

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Chapter 1
Introduction

1.1 Motivation

From the simplest cellphone charger to the most complex bipedal robot, the use of digital control systems incorporating decision-making algorithms which act upon continuous-time processes is pervasive. This interaction between systems of continuous nature at a lower level, and discrete events at a higher level is subsumed under the term *hybrid systems*. For example, the switching of a transistor in a power converter, an automotive transmission changing gears, a thermostat turning the heat on or off, or an aircraft flight control system alternating between different flight modes are some of the many real-life processes that exhibit an intrinsically hybrid behavior. Motivated by the remarkable advancement of embedded systems over these past decades, and its application in control systems, the study of hybrid systems has never had more relevance than it does now. Although in widespread use today, the history of hybrid systems is fairly recent, stemming from the introduction and rapid adoption of the electromagnetic relay in industrial automation by the mid 1900's. Since then, progressively more complicated hybrid systems displaying unique behaviors have emerged, and with them, the necessity to develop tools and techniques for the modeling, analysis and control of these types of systems, taking into account the intertwined nature of the continuous and discrete dynamics. Further details on this topic can be found in [1, 2].

A particularly important subclass of hybrid systems, known as switched systems, has gathered much attention lately due to its usefulness in modeling a wide range of applications. For instance, active and semi-active automotive suspensions control [3, 4], the control of wind turbines [5], power electronics [6, 7], aircraft roll angle control [8], and autonomous robotics [9, 10]. In essence, these systems are comprised of a finite set of subsystems, defining the available modes of operation, and a switching signal that selects which mode will be active at each instant of time. As such, these systems exhibit a complex and nonlinear dynamical behavior, distinct to that of their modes of operation. Furthermore, a suitable switching signal may stabilize the switched system, in the case where all modes of operation are unstable, or conversely, destabilize a switched system even when all subsystems are stable. This underscores the importance of the switching signal, which can be an arbitrary time-dependent signal, such as an external input or disturbance, or a control variable to be designed. In both cases, the control problem is centered on developing conditions for stability, and additionally, assuring certain performance criteria for the switched system. The books [11, 12] and the survey [13] explore this topic in greater detail.

These subsystems may individually present different kinds of dynamical behavior. In this work, the focus is given to switched systems comprised of affine subsystems, referred to as switched affine systems. This case is more general when compared to a switched system composed of linear subsystems, and introduces greater difficulty by giving rise to a set of attainable nontrivial equilibrium points. This particular characteristic brings light to the relevance of this category of switched systems, especially given the multitude of applications that can be modeled in this framework. One such application is the previously mentioned switched mode DC-DC power converter, and switched affine systems are especially suited to model these electronic circuits, given their switched nature, and affine dynamical behavior. These circuits are ubiquitous, and power nearly all electronic devices that operate from a DC source. As such, several publications in the literature treat of the *buck*, *buck-boost* and *flyback* topologies [14, 15, 16, 17, 18], and are concerned with devising an appropriate switching function tasked with maintaining a certain operating condition.

Many results regarding the design of an appropriate switching function, also referred to as a switching rule, in order to guarantee asymptotic stability of switched linear system are already solidly established [19, 20, 21, 22, 23, 15, 24, 25] some of which also deal with performance criteria, such as the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ indices, and will be presented in greater detail in Chapter 3. With regard to switched affine systems, some references consider exclusively the action of a switching function dependent on state information, capable of stabilizing the system [26, 27, 28], while others deal with the joint action of a switching function, alongside a control law [29]. More specifically, the latter is based on a set of state feedback matrix gains and a state dependent switching rule assuring asymptotic stability of the closed-loop system, while guaranteeing optimal $\mathcal{H}_2$ or $\mathcal{H}_\infty$ costs.

In practical applications, however, it is not uncommon for some state variables to be unavailable, whether due to the difficulty or expense in effecting these measurements; because of physical constraints in sensor placement or size; or simply due to the unavailability of sensors to measure a certain state variable. As such, control design methodologies that rely on output information in place of state information constitute a very relevant and applicable topic of study. Existing results in the literature consider the case where the only control variable is an output-dependent switching rule [30, 16] implemented via a switched dynamical filter responsible for providing the information needed. On the other hand, as of yet, results in the literature treating the control design problem considering the joint action of an output dependent stabilizing switching function and a dynamical controller, do so only for switched linear systems [24]. To fill this void in the context of switched affine systems, in this work we propose three control design techniques based on a full-order switched dynamical controller, also available in [31]. The first generalizes the results of [24], taking into account an output-dependent switching rule and control law that together guarantee global asymptotic stability of a desired equilibrium point for the closed-loop system, as well as an upper bound for the $\mathcal{H}_\infty$ performance index. The second reduces the conservativeness of these conditions by considering a switching rule dependent on both output and input information. The third technique introduces an output-dependent switching rule and control law assuring an $\mathcal{H}_2$ guaranteed cost and global asymptotic stability of the equilibrium point of interest. The proposed approaches do not require any stability property of the individual subsystems, guaranteeing stability when the subsystems are not controllable, or when a stable convex combination of dynamical matrices cannot be verified. Furthermore, the proposed techniques allow for a wider range of equilibrium points to be attained when compared to existing methods. This reinforces the importance of the joint action of both control structures, and will be discussed at length in Chapter 4.

This work in its present form, encompasses part of my Master's dissertation, which besides the contents here presented, will also study the classical filtering problem for switched systems. To the best of our knowledge, this problem has yet to be treated in the context of switched affine systems. The following publications represent the results obtained during my Master's program.

1.2 Publications

This monograph is based in part on the following papers:

- G. K. Kolotelo and G. S. Deaecto, "Controle $\mathcal{H}_2$ e $\mathcal{H}_\infty$ via Realimentação de Saída de Sistemas Afins com Comutação por Ação Conjunta de Função de Comutação e Entrada de Controle", Congresso Brasileiro de Automática - CBA, *Submitted*.
- G. K. Kolotelo, L. N. Egidio, and G. S. Deaecto, "$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Filtering for Continuous–Time Switched Affine Systems", IFAC Symposium on Robust Control Design ROCOND, 2018, *To appear*.
- G. K. Kolotelo, L. N. Egidio, and G. S. Deaecto, "Projeto de Filtros com Comutação $\mathcal{H}_2$ e $\mathcal{H}_\infty$ para Sistemas Afins a Tempo Contínuo", Congresso Brasileiro de Automática - CBA, *Submitted*.

1.3 Outline of Chapters

This work is divided in 5 chapters, explained in brief:

- **Chapter 1: Introduction**

  Presents the motivation and sets the context for the topics that are treated in this work.
- **Chapter 2: Preliminaries**

  Reviews the fundamental concepts of dynamical systems, important for the next chapters. In particular, the stability properties of dynamical systems are studied via Lyapunov's stability theory. Lastly, performance criteria for these systems are defined.
- **Chapter 3: Switched Systems**

  Broaches the subject of switched systems, and discusses in greater detail their unique features. Next, well known results in the literature are introduced, which present conditions for the stability of switched linear and affine systems. Finally, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices for switched affine systems are defined, important for the subsequent chapters.
- **Chapter 4: Joint Action Output Feedback Control**

  Presents the main results of this monograph in regard to the joint design of an output-dependent switching rule alongside a control law. More specifically, the methodology for the design of a full-order switched dynamical controller and a switching function that together assure global asymptotic stability of a desired equilibrium point, as well as a guaranteed $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance indices is introduced. A set of numerical examples are provided to illustrate the theory.

- **Chapter 7: Conclusion**

  Summarizes the topics explored by this dissertation, and examines some prospects for future developments on these topics.

Chapter 2
Preliminaries

The analysis of stability properties in the context of switched systems is largely based on the theory of stability introduced by Lyapunov, due to the nonlinear behavior that is intrinsic to these types of systems. As such, prior to delving into this subject, we review some key ideas and definitions that permeate this work, and constitute the theoretical basis under which its results depend.

The purpose of this chapter is to introduce the underlying concepts concerning the stability analysis and performance indices for Linear Time-Invariant systems, henceforth referred to as a LTI system. Firstly, following a brief discussion on the concepts of equilibrium and stability, Lyapunov's stability theory is introduced. More specifically, the second method of Lyapunov, also known as the direct method, extensively used in the analysis and control of nonlinear systems, will be used to verify the conditions for which stability properties of a dynamical system are verified for a certain equilibrium point. Lastly, the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for LTI systems will be presented. These ideas are extremely important in classical control theory, and will be extended to deal with the stability analysis and the control design problem for switched systems, considered in the next chapters. References [32, 33, 34] will be used to support the discussions throughout.

2.1 Stability of LTI Systems

Before investigating the stability properties of dynamical systems in the following sections, let us first introduce some basic concepts regarding linear dynamical systems. The state space representation of a continuous-time LTI system is given by

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{H}\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{w}(t) \\
\mathbf{z}(t) &= \mathbf{E}\mathbf{x}(t) + \mathbf{F}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(t)
\end{aligned} \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, with the number of states n_x being referred to as the order of the system, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the disturbance, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ is the measured output, and $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance output. Also $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are constant matrices of appropriate dimensions. In the case of $\mathbf{G} = \mathbf{0}$, the system is named a strictly proper system. For simplicity, we initiate our discussions considering the following unforced LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.2}$$

which disregards the inputs and outputs of system (2.1), allowing us to analyze its stability properties.

2.1.1 Equilibrium Points

A state vector $\mathbf{x}_e$ is termed an equilibrium point of the system, if once $\mathbf{x}(t) = \mathbf{x}_e$, for some $t = t_0$, the system remains at the equilibrium point from that moment onwards, or equivalently, $\dot{\mathbf{x}}(t) = \mathbf{0}$ for $t \geq t_0$. For the case of linear systems, the origin $\mathbf{x}_e = \mathbf{0}$ will always be an equilibrium point. Moreover, it will be the single equilibrium point of the system, unless $\mathbf{A}$ is singular, in which case, there exist multiple equilibrium points beyond $\mathbf{x}_e = \mathbf{0}$, given by the null space of the matrix $\mathbf{A}$, such that $\mathbf{A}\mathbf{x} = \mathbf{0}$. See [32, 33] for more details.

For convenience, any equilibrium point can be shifted to the origin by means of the change of variables $\xi = \mathbf{x} - \mathbf{x}_e$ with no loss of generality, see [33, 34], facilitating some of our subsequent analyses.

2.1.2 Stability of Dynamical Systems

The concept of stability for dynamical systems is defined in terms of an equilibrium point. More specifically, [32] defines the equilibrium point $\mathbf{x}_e \in \mathbb{R}^{n_x}$ as stable if whenever the state vector is moved slightly away from that point, it tends to return to it, or at least does not keep moving further away. In other words, the state vector remains within a bounded region in the state space around the equilibrium point. Furthermore, if whenever $t \to \infty$, the state vector $\mathbf{x}(t) \to \mathbf{x}_e$, then the equilibrium point is referred to as asymptotically stable. It is designated unstable, if for a given initial condition$\mathbf{x}(0)$, the state vector $\mathbf{x}(t)$ tends to infinity. Whenever this concept of stability only holds for initial conditions within a certain region of the state space, the equilibrium point is said to be locally stable. If however, it is valid for for any given initial condition, it is said to be globally stable. Moreover, the equilibrium point is globally asymptotically stable if $\mathbf{x}(t) \to \mathbf{x}_e$ as $t \to \infty$, for any initial condition.

For the specific case of LTI systems, necessary and sufficient conditions for the stability of continuous-time systems can be derived by studying the eigenvalues of the matrix $\mathbf{A}$. For these systems, an equilibrium point is globally asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ are located at the open left half complex plane, that is, the eigenvalues have strictly negative real part. In this case, the matrix $\mathbf{A}$ is said to be Hurwitz, or $\mathbf{A} \in \mathcal{H}$, where $\mathcal{H}$ is defined as the set of Hurwitz matrices.

2.2 Lyapunov Stability Theory

For many decades, one of the most useful techniques for the evaluation of stability properties for linear and nonlinear systems has been the theory introduced by Lyapunov in the late 1800's. When compared to linear systems, nonlinear systems may exhibit entirely new and complex behaviors. This, coupled with the fact that explicit analytical solutions most often cannot be attained for these systems, sheds light upon the importance of Lyapunov's findings. The theory is comprised of two widely used methods, commonly referred to as the linearization or indirect method, and the direct method. The latter will be used extensively throughout this work, and will be introduced below for LTI systems, followed by a brief discussion of its application on affine nonlinear systems.

2.2.1 Lyapunov's Direct Method

Lyapunov's direct method, as defined in [32, 33, 34], can be used to evaluate the stability of linear and nonlinear systems in an indirect manner, enabling the characterization of the stability properties of all equilibrium points for a given system, without the need to derive explicit numerical or analytical solutions. The stability analysis is based on a scalar *energy-like* or *summarizing* function, dependent on the state vector. The purpose of this function, known as a Lyapunov function, is to describe the behavior of the entire dynamical system as a function of time, and allow conclusions to be inferred on the stability of the set of its governing differential equations. The search for a suitable Lyapunov function may be a difficult task for more sophisticated systems, since it must respect the set of requirements introduced below.

2.2.1.1 Lyapunov Function

A scalar function $v : \mathbb{D} \rightarrow \mathbb{R}$, with $\mathbb{D} \subset \mathbb{R}^{n_x}$ a region of the state space containing $\mathbf{x} = \mathbf{0}$, is a Lyapunov function if it satisfies the following set of conditions [32, 34]:

1. v is continuously differentiable
2. $v(\mathbf{0}) = 0$
3. $v(\mathbf{x}(t)) > 0, \quad \forall \mathbf{x} \in \mathbb{D},\ \mathbf{x} \neq \mathbf{0}$
4. $\dot{v}(\mathbf{x}(t)) \leq 0, \quad \forall \mathbf{x} \in \mathbb{D}$

If these requirements are verified for the system in question, then the equilibrium point $\mathbf{x}_e = \mathbf{0}$ is stable. Moreover, whenever

5. $\dot{v}(\mathbf{x}(t)) < 0, \quad \forall \mathbf{x} \in \mathbb{D},\ \mathbf{x} \neq \mathbf{0}$

is satisfied, then $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable. Additionally, if

6. $\|\mathbf{x}\| \rightarrow \infty \;\; \Rightarrow \;\; v(\mathbf{x}) \rightarrow \infty, \quad \mathbb{D} = \mathbb{R}^{n_x}$ *radially unbounded*

can also be verified, then the equilibrium point at the origin is globally asymptotically stable.

2.2.1.2 Lyapunov Stability Analysis of LTI Systems

For LTI systems, an adequate choice for the Lyapunov function v is the quadratic form of a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n_x \times n_x}$ such that

$$v(\mathbf{x}(t)) = \mathbf{x}(t)^T \mathbf{P} \mathbf{x}(t) \tag{2.3}$$

It can be verified that this function meets all the conditions of a Lyapunov function. Indeed, conditions (1) – (3) are evidently verified. In addition, (6) is satisfied, and the validity of condition (5) can be demonstrated as follows

$$\begin{aligned} \dot{v}(\mathbf{x}(t)) &= \dot{\mathbf{x}}(t)^T\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^T\mathbf{P}\dot{\mathbf{x}}(t) \\ &= \mathbf{x}(t)^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x}(t) \\ &= -\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) \end{aligned} \tag{2.4}$$

where $\mathbf{Q} \in \mathbb{R}^{n_x \times n_x}$ is any given symmetric positive definite matrix. In this case, if there exists $\mathbf{P} \succ 0$ satisfying the so-called Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0 \tag{2.5}$$

for a given $\mathbf{Q} \succ 0$, then $\dot{v}(\mathbf{x}(t)) = -\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) < 0\ \forall \mathbf{x}(t) \neq \mathbf{0}$ is valid, assuring the equilibrium point $\mathbf{x}_e = \mathbf{0}$ of the system to be globally asymptotically stable.

It should be mentioned that the existence of a positive definite solution $\mathbf{P} \succ 0$ for the Lyapunov equation is not only a sufficient condition for global asymptotic stability, but also a necessary condition. Indeed, assume that matrix $\mathbf{A}$ is Hurwitz, and consider a symmetric positive definite matrix $\mathbf{Q}$, as well as a matrix $\mathbf{P}$ defined by

$$\mathbf{P} = \int_0^\infty \mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t}\,\mathrm{d}t \tag{2.6}$$

Whenever the eigenvalues of $\mathbf{A}$ have strictly negative real part, the integral exists. Furthermore $\mathbf{P}$ is symmetric positive definite since, for any vector $\mathbf{u} \neq \mathbf{0}$

$$\begin{aligned} \mathbf{u}^T\mathbf{P}\mathbf{u} &= \int_0^\infty \mathbf{u}^T\mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t}\mathbf{u}\,\mathrm{d}t \\ &= \int_0^\infty \mathbf{x}^T\mathbf{Q}\mathbf{x}\,\mathrm{d}t \end{aligned} \tag{2.7}$$

where $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{u}$ is the analytical solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, assigning $\mathbf{u} = \mathbf{x}(0)$. As $\mathbf{Q} \succ 0$, then $\mathbf{P}$ must be positive definite. Finally, by substituting (2.6) in (2.5) we have

$$\begin{aligned} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} &= \int_0^\infty \mathbf{A}^T\mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t}\,\mathrm{d}t + \int_0^\infty \mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t}\mathbf{A}\,\mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t}\,\mathrm{d}t \\ &= \lim_{t\to\infty}\mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t} - \mathbf{Q} \\ &= -\mathbf{Q} \end{aligned} \tag{2.8}$$

which indicates that whenever matrix $\mathbf{A}$ is Hurwitz, and thus all eigenvalues its have strictly negative real part, then $\lim_{t\to\infty}\mathrm{e}^{\mathbf{A}^T t}\mathbf{Q}\mathrm{e}^{\mathbf{A}t} = \mathbf{0}$, and the Lyapunov equation is verified.

This gives rise to the following theorem, prescribing asymptotic stability of the origin for LTI systems, in terms of the solution of the Lyapunov equation, stated in [34] as follows

Theorem 2.1. *A matrix* $\mathbf{A}$ *is Hurwitz, that is, the eigenvalues of* $\mathbf{A}$ *have strictly negative real part if and only if, for* *any given symmetric positive definite matrix* **Q**, *there exists a symmetric positive definite matrix* **P** *that satisfies the Lyapunov equation* (2.5).

It is worth noting that although LTI system stability can be easily determined by evaluating the eigenvalues of the matrix **A**, this analysis is rendered moot when studying other types of systems, including switched systems, which will be presented in more detail in the ensuing chapters. This reinforces the influence of Lyapunov's theory for the stability analysis of complex and nonlinear systems.

Finally, a related point to consider is that the Lyapunov equation was originally expressed in terms of linear matrix inequalities, in the form known as the Lyapunov inequality

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} \prec 0 \tag{2.9}$$

This inequality was originally solved for **P** analytically, via a set of linear equations, by choosing any $\mathbf{Q} \succ 0$ that satisfied (2.5), as discussed in [35]. Only in the 1980's it would become clear that the Lyapunov inequality could be solved numerically by means of convex optimization algorithms, with great efficiency. Throughout this work, we will focus on expressing stability conditions and control design problems in terms of Linear Matrix Inequalities, from this point onwards referred to as LMIs, which can be solved without difficulty by several standard and readily available tools.

2.2.1.3 Lyapunov Stability Analysis of Affine Systems

The analysis of stability in the context of switched affine systems is central to the topics explored in this work. As such, it is important to initially study the stability properties of the affine subclass of nonlinear systems. To this extent, consider the following affine system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.10}$$

where $\mathbf{b} \in \mathbb{R}^{n_x}$ is the affine term. Notice that whenever $\mathbf{b} = \mathbf{0}$, this system reduces to the linear case, previously presented. Also, recall that by the definition of an equilibrium point, we have $\dot{\mathbf{x}}(t) = \mathbf{0}$. Thus, the equilibrium point of the affine system can be calculated as

$$\mathbf{x}_e = -\mathbf{A}^{-1}\mathbf{b} \tag{2.11}$$

As previously mentioned, in order to simplify evaluating the stability properties of this system by means of Lyapunov's direct method, and incurring no loss of generality, the state vector $\boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_e$ is defined, and we now consider the system (2.10) shifted to the origin, as such

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{A}\boldsymbol{\xi}(t), \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \tag{2.12}$$

Observe that the problem can now be treated as an LTI system, such that the methodology and conditions for stability introduced in Section 2.2.1.2 remain valid for the analysis of affine time invariant systems. In this case, for a certain affine system under consideration, whenever there exists $\mathbf{P} \succ 0$ such that the Lyapunov inequality (2.9) is satisfied for this system, then $\mathbf{x}_e$ is as globally asymptotically stable equilibrium point, since as $\boldsymbol{\xi} \rightarrow \mathbf{0}$, we have $\mathbf{x} \rightarrow \mathbf{x}_e$.

2.3 Performance Indices for LTI Systems

In this section the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for LTI systems are introduced. These norms are used extensively in control design, see [34, 35, 36, 37], to characterize the effects of a given input signal on the output of the system. They will be expressed both with respect to the transfer function of the system, as well as in terms of its impulse response, the latter being essential to allow their generalization in order to deal with switched systems, thus providing an effective measure of performance for this class of systems. But first, some fundamental concepts that are important to the development of the next topics will be presented, namely, Parseval's Theorem for continuous-time LTI systems, and the $\mathcal{L}_2$ space, will be introduced.

2.3.1 Parseval's Theorem

Consider the function $\mathbf{f}(t) : [0,\infty) \rightarrow \mathbb{R}^n$ and its Laplace transform $\mathbf{F}(s)$, as well as the conjugate transpose $\mathbf{F}(s)^*$, whose domain dom($\mathbf{F}$), contains the imaginary axis. Parseval's theorem is then defined by [38] as

$$\begin{aligned} \int_0^\infty \mathbf{f}(t)^T \mathbf{f}(t)\, \mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{F}(\mathrm{j}\omega)^* \mathbf{F}(\mathrm{j}\omega) \mathrm{d}\omega \\ &= \frac{1}{\pi} \int_0^{\infty} \mathbf{F}(\mathrm{j}\omega)^* \mathbf{F}(\mathrm{j}\omega) \mathrm{d}\omega \end{aligned} \tag{2.13}$$

This relation will later be employed for the calculation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms in the coming sections.

2.3.2 $\mathcal{L}_2$ Space

The norm $||\cdot|| : \mathbb{R}^n \rightarrow \mathbb{R}$, as defined in [34, 39], is a real-valued function satisfying the following four axioms, for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and all $\alpha \in \mathbb{R}$:

- $||\mathbf{v}|| \geq 0$ *Nonnegativity*
- $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = 0$ *Positivity*
- $||\alpha \mathbf{v}|| = |\alpha| ||\mathbf{v}||$ *Homogeneity*
- $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ *Triangle Inequality*

One specific example, the Euclidean norm, or $\mathcal{L}_2$ norm, defined for any vector $\mathbf{x} \in \mathbb{R}^n$

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \left(\mathbf{x}^T\mathbf{x}\right)^{1/2} \tag{2.14}$$

is perhaps the most commonly used norm. The definition of a norm, however, is not exclusive to finite-dimensional vector spaces. It is also useful to define the $\mathcal{L}_2$ norm for continuous real-valued functions of the form $\mathbf{f}(t)\colon [0,\infty) \to \mathbb{R}^n$, as such

$$\|\mathbf{f}\|_2 = \left(\int_0^\infty \|\mathbf{f}(t)\|_2^2 \,\mathrm{d}t\right)^{1/2} = \left(\int_0^\infty \mathbf{f}(t)^T\mathbf{f}(t)\,\mathrm{d}t\right)^{1/2} \tag{2.15}$$

If the integral amounts to a finite value, the function $\mathbf{f}(t)$ is called a square-integrable function. This characterization of the norm for a function will be helpful to measure the magnitude of the input and output signals of a dynamical system, allowing for the definition of the performance criteria introduced in the next section.

2.3.3 System Definition

The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are introduced considering the following LTI system, defined for $t \geq 0$ with $\mathbf{A}$ Hurwitz.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{H}\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{0} \\ \mathbf{z}(t) &= \mathbf{E}\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t) \end{aligned} \tag{2.16}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the disturbance, $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance output, and matrices $\mathbf{A}, \mathbf{H}, \mathbf{E}, \mathbf{G}$ are of appropriate dimensions. The transfer matrix $\mathbf{H}_{wz}(s) \in \mathbb{R}^{n_z \times n_w}$ of system (2.16), from the input $\mathbf{w}$ to the output $\mathbf{z}$ is then given by

$$\mathbf{H}_{wz}(s) = \mathbf{E}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{H} + \mathbf{G} \tag{2.17}$$

with $s \in \mathbb{C}$. We can now proceed to the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for continuous-time LTI systems.

2.3.4 $\mathcal{H}_2$ norm for LTI systems

The $\mathcal{H}_2$ norm can be interpreted as a measure of the energy of the output signal of a dynamical system, when driven by an impulse. Other interpretations, such as in the context of stochastic systems exist, but will not be discussed in this work. For the continuous-time LTI system (2.16), the $\mathcal{H}_2$ norm may be calculated whenever a strictly proper transfer matrix $\mathbf{H}_{wz}(s)$ is considered, that is, $\mathbf{G} = \mathbf{0}$. In this case, the $\mathcal{H}_2$ norm is defined by [40] as

$$\|\mathbf{H}_{wz}(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(\mathbf{H}_{wz}(\mathrm{j}\omega)^*\mathbf{H}_{wz}(\mathrm{j}\omega)\right)\mathrm{d}\omega \tag{2.18}$$

By applying Parseval's Theorem, introduced in (2.13), the $\mathcal{H}_2$ norm can be expressed in the time domain, as such

$$||\mathbf{H}_{wz}(s)||_2^2 = \int_0^\infty \mathrm{tr}\left(\mathbf{h}_{wz}(t)^T \mathbf{h}_{wz}(t)\right)\,\mathrm{d}t \tag{2.19}$$

By realizing that the impulse response $\mathbf{h}_{wz}(t)$ of the system, when initial conditions are set to zero, is

$$\mathbf{h}_{wz}(t) = \begin{cases} \mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathbf{H}, & t \geq 0 \\ \mathbf{0}, & \text{otherwise} \end{cases} \tag{2.20}$$

and that for multiple-input, multiple-output systems, $\mathbf{h}_{wz}(t)$ is of the form

$$\mathbf{h}_{wz}(t) = \begin{bmatrix} h_{11}(t) & \dots & h_{n_w}(t) \\ \vdots & \ddots & \vdots \\ h_{n_z}(t) & \dots & h_{n_z n_w}(t) \end{bmatrix} \tag{2.21}$$

then, from (2.19) and (2.21), the $\mathcal{H}_2$ norm can be written as

$$||\mathbf{H}_{wz}(s)||_2^2 = \sum_{k=1}^{n_w}\sum_{i=1}^{n_z}\int_0^\infty h_{ik}^2(t)\,\mathrm{d}t \tag{2.22}$$

It is worth noting that for single-input, single-output systems, the $\mathcal{H}_2$ norm becomes simply the $\mathcal{L}_2$ norm of the impulse response for the system in question. This emphasizes the need of considering a strictly proper system in order to obtain a finite $\mathcal{H}_2$ norm. Furthermore, observe that equation (2.22) may alternatively be expressed as

$$||\mathbf{H}_{wz}(s)||_2^2 = \int_0^\infty \mathrm{tr}\left(\mathbf{H}^T \mathrm{e}^{\mathbf{A}^T t}\mathbf{E}^T\mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathbf{H}\right)\,\mathrm{d}t = \int_0^\infty \mathrm{tr}\left(\mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathbf{H}\mathbf{H}^T\mathrm{e}^{\mathbf{A}^T t}\mathbf{E}^T\right)\mathrm{d}t \tag{2.23}$$

This allows the $\mathcal{H}_2$ norm to be stated either as

$$||\mathbf{H}_{wz}(s)||_2 = \sqrt{\mathrm{tr}\left(\mathbf{H}^T\mathbf{L}_o\mathbf{H}\right)}, \qquad \text{or} \qquad ||\mathbf{H}_{wz}(s)||_2 = \sqrt{\mathrm{tr}\left(\mathbf{E}\mathbf{L}_c\mathbf{E}^T\right)} \tag{2.24}$$

where the matrices $\mathbf{L}_o$ and $\mathbf{L}_c$ are respectively referred to as the controllability Gramian and the observability Gramian, given by

$$\mathbf{L}_o = \int_0^\infty \mathrm{e}^{\mathbf{A}^T t}\mathbf{E}^T\mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathrm{d}t, \qquad \text{and} \qquad \mathbf{L}_c = \int_0^\infty \mathrm{e}^{\mathbf{A}t}\mathbf{H}\mathbf{H}^T\mathrm{e}^{\mathbf{A}^T t}\mathrm{d}t \tag{2.25}$$

These are, in turn, the solutions to their associated Lyapunov equations, briefly discussed in Section 2.2.1.2, as follows

$$\mathbf{A}^T\mathbf{L}_o + \mathbf{L}_o\mathbf{A} + \mathbf{E}^T\mathbf{E} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T + \mathbf{H}\mathbf{H}^T = \mathbf{0} \tag{2.26}$$

Observe that the $\mathcal{H}_2$ norm, expressed in this manner, can be easily solved via numerical methods by the following convex optimization problem, subject to LMI restrictions, as shown in [35]

$$\begin{aligned} \min \quad & \operatorname{tr}\left(\mathbf{H}^T\mathbf{P}\mathbf{H}\right) \\ \text{subject to:} \quad & \mathbf{P} \succ 0 \\ & \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}\mathbf{E}^T \prec 0 \end{aligned} \tag{2.27}$$

$$\begin{aligned} \min \quad & \operatorname{tr}\left(\mathbf{E}\mathbf{P}\mathbf{E}^T\right) \\ \text{subject to:} \quad & \mathbf{P} \succ 0 \\ & \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{H}\mathbf{H}^T \prec 0 \end{aligned} \tag{2.28}$$

It is important to note that the 'min' and 'inf' terms for optimization problems can be used interchangeably whenever we consider that the non-compact set of restrictions is closed by the numerical solver to a known tolerance $\epsilon > 0$. As such, the solution $\mathbf{P}$ obtained is arbitrarily close to the respective solutions of $\mathbf{L}_c$ or $\mathbf{L}_o$ in (2.25). In this case, the $\mathcal{H}_2$ norm is given by $\|\mathbf{H}_{wz}(s)\|_2^2 = \operatorname{tr}\left(\mathbf{H}^T\mathbf{L}_o\mathbf{H}\right) < \operatorname{tr}\left(\mathbf{H}^T\mathbf{P}\mathbf{H}\right)$, or alternatively $\|\mathbf{H}_{wz}(s)\|_2^2 = \operatorname{tr}\left(\mathbf{E}\mathbf{L}_c\mathbf{E}^T\right) < \operatorname{tr}\left(\mathbf{E}\mathbf{P}\mathbf{E}^T\right)$.

2.3.5 $\mathcal{H}_\infty$ norm for LTI systems

The $\mathcal{H}_\infty$ norm characterizes a measure of greatest possible $\mathcal{L}_2$ gain of the system, which is the ratio between the $\mathcal{L}_2$ norm of the output signal and the $\mathcal{L}_2$ norm of a square integrable input signal, across any input channel, that maximizes this ratio. It is defined for system (2.16), considering external inputs $\mathbf{w} \in \mathcal{L}_2$, and is defined by [34] as

$$\|\mathbf{H}_{wz}(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \|\mathbf{H}_{wz}(\mathrm{j}\omega)\|_2 \tag{2.29}$$

For single-input, single-output systems, the $\mathcal{H}_\infty$ norm becomes simply the peak gain observed for the frequency response of $\mathbf{H}_{wz}(\mathrm{j}\omega)$, $\omega \in \mathbb{R}$.

The $\mathcal{H}_\infty$ norm can also be defined in the time domain, as demonstrated in [34, 41], by the following

$$\|\mathbf{H}_{wz}(s)\|_\infty = \sup_{0\neq\mathbf{w}\in\mathcal{L}_2} \frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2} \leq \gamma \tag{2.30}$$

where the scalar $\gamma > 0$ is the upper bound of the $\mathcal{H}_\infty$ norm of system (2.16). Alternatively, (2.30) can be rewritten, when considering (2.15), as

$$\int_0^\infty \mathbf{z}(t)^T\mathbf{z}(t)\,\mathrm{d}t \leq \gamma^2 \int_0^\infty \mathbf{w}(t)^T\mathbf{w}(t)\,\mathrm{d}t, \quad \mathbf{w}(t) \neq 0, \quad \mathbf{w}(t) \in \mathcal{L}_2, \quad t \geq 0 \tag{2.31}$$

This definition allows the $\mathcal{H}_\infty$ norm to be expressed completely in terms of the input and output signals in the time domain. When considering performance indices for switched systems, this becomes especially important, since these systems cannot be expressed in terms of transfer functions, as will become clear in the forthcoming chapter.

As for the $\mathcal{H}_2$ norm, the upper bound for $\mathcal{H}_\infty$ norm can also be calculated by means of a convex optimization problem subject to LMI constraints. By considering the quadratic Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$, with $\mathbf{P} \succ 0$, we have

$$\begin{aligned}\dot{v}(\mathbf{x}) &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} + (\mathbf{z}^T\mathbf{z} - \mathbf{z}^T\mathbf{z}) + (\gamma^2\mathbf{w}^T\mathbf{w} - \gamma^2\mathbf{w}^T\mathbf{w}) \\ &= \begin{bmatrix}\mathbf{x}\\ \mathbf{w}\end{bmatrix}^T \begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} & \mathbf{P}\mathbf{H} + \mathbf{E}^T\mathbf{G} \\ \mathbf{H}^T\mathbf{P} + \mathbf{G}^T\mathbf{E} & \mathbf{G}^T\mathbf{G} - \gamma^2\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{w}\end{bmatrix} - \mathbf{z}^T\mathbf{z} + \gamma^2\mathbf{w}^T\mathbf{w} \\ &< -\mathbf{z}^T\mathbf{z} + \gamma^2\mathbf{w}^T\mathbf{w}\end{aligned} \tag{2.32}$$

where the inequality arises by imposing

$$\begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} & \bullet \\ \mathbf{H}^T\mathbf{P} + \mathbf{G}^T\mathbf{E} & \mathbf{G}^T\mathbf{G} - \gamma^2\mathbf{I}\end{bmatrix} \prec 0 \tag{2.33}$$

Notice that a necessary condition for the feasibility of this inequality is that $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} \prec 0$, thus implying in $\dot{v}(\mathbf{x}) < 0$, reinforcing the fact that the system is asymptotically stable.

By integrating both sides of (2.32) from $t = 0$ to $t \to \infty$, we have

$$\begin{aligned}\int_0^\infty \dot{v}(\mathbf{x}(t))\, \mathrm{d}t &< \int_0^\infty -\mathbf{z}^T\mathbf{z} + \gamma^2\mathbf{w}^T\mathbf{w}\, \mathrm{d}t \\ \lim_{t\to\infty} v(\mathbf{x}(t)) - v(\mathbf{x}(0)) &< -\|\mathbf{z}\|_2^2 + \gamma^2\|\mathbf{w}\|_2^2\end{aligned} \tag{2.34}$$

which becomes (2.30), since $\lim_{t\to\infty} v(\mathbf{x}(t)) = 0$, as the system is globally asymptotically stable, and $v(\mathbf{x}(0)) = 0$ given that $\mathbf{x}(0) = \mathbf{0}$. As such, whenever (2.33) is satisfied, the inequality (2.32) is guaranteed, and by consequence, so is (2.30). By means of a convex optimization problem, described in terms of LMIs, the $\mathcal{H}_\infty$ norm can be calculated with no difficulty, as follows

$$\begin{aligned} \min \quad & \rho \\ \text{subject to:} \quad & \mathbf{P} \succ 0,\ \rho > 0 \\ & \begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} & \bullet & \bullet \\ \mathbf{H}^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E} & \mathbf{G} & -\mathbf{I}\end{bmatrix} \prec 0\end{aligned} \tag{2.35}$$

where the last inequality is equivalent to (2.33), made clear by applying Schur complement with respect to $-\mathbf{I}$. The $\mathcal{H}_\infty$ norm is then given by $\|\mathbf{H}_{wz}(s)\|_\infty < \gamma$, with $\rho = \gamma^2$. Again, as in the $\mathcal{H}_2$ case, the solution ρ is arbitrarily close to the analytical solution, only by the specified tolerance $\epsilon > 0$ of the numerical solver.

Chapter 3

Switched Systems

In this chapter, we introduce the concept of switched systems, followed by the study of the stability properties of switched linear and affine systems, as well as revelant performance criteria, which generalize the concepts previously established. The topics presented in this chapter review several foundational results already existent in the literature, for instance [13, 19] for switched linear systems, and [15, 29] for switched affine system, as well as the books [12] and [11], important to support the ideas presented later in this work.

3.1 Introduction

Switched systems constitute a subclass of hybrid systems, in the sense that these systems are governed by a set of modes of operation, each of which may be represented by a dynamical system, and is coupled with discrete switching events across these modes, thus affecting the trajectory of the overall system. The switching between modes is orchestrated by a switching function, also known as a switching rule, denoted by $\sigma(\cdot)$. It encompasses a decision-making process that selects values within the set $\mathbb{K} := \{1, \ldots, N\}$, at every instant of time, such that each $i \in \mathbb{K}$ corresponds to an individual mode of operation, refereed to as a subsystem of the switched system. This chapter will deal with the so-called continuous-time switched linear systems, which concerns the case where all subsystems are governed by linear dynamical systems, and subsequently, we discuss the continuous-time switched affine systems, pertaining to the situation where at least one of the subsystems presents a nonzero affine term, contemplating the main focus of this work.

The effect of switching in switched systems is not trivial, since not only does it establish the nonlinear and time-varying nature of these systems, but also, the stability properties of the switched system are inherently dependent on the switching signal. Indeed, it may give rise to complex and unprecedented behaviors, even when simple subsystems are considered. An example of this is the occurrence of sliding modes, in which the switched system switches infinitely fast. This specific situation, although sometimes undesirable, allows for a behavior significantly different than that of each isolated subsystem, and in the case of switched affine systems, it introduces new attainable equilibrium points that are distinct from those of each subsystem, a topic that will be discussed in greater detail shortly. It is important to note that although the switching function modifies the trajectory of the switched system, the continuous state evolves without discontinuities, that is, the state does not jump impulsively on switching events.

To illustrate how the stability properties of a switched system are intertwined with the switching signal, consider two unforced stable linear subsystems given by $\dot{\mathbf{x}} = \mathbf{A}_i\mathbf{x}$, for $i = \{1, 2\}$. Individually, these subsystems would display a monotonically decreasing Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}_i\mathbf{x}$, however, when subject to a specific switching signal, the switched system may prove to be unstable, as exemplified in Figure 3.1. In this figure it can be observed that although $v(\mathbf{x}(t))$ decreases while the subsystem is active, an upward trend for $v(\mathbf{x}(t))$ can be seen, indicating that the switching signal under consideration destabilizes the switched system. Fortunately,

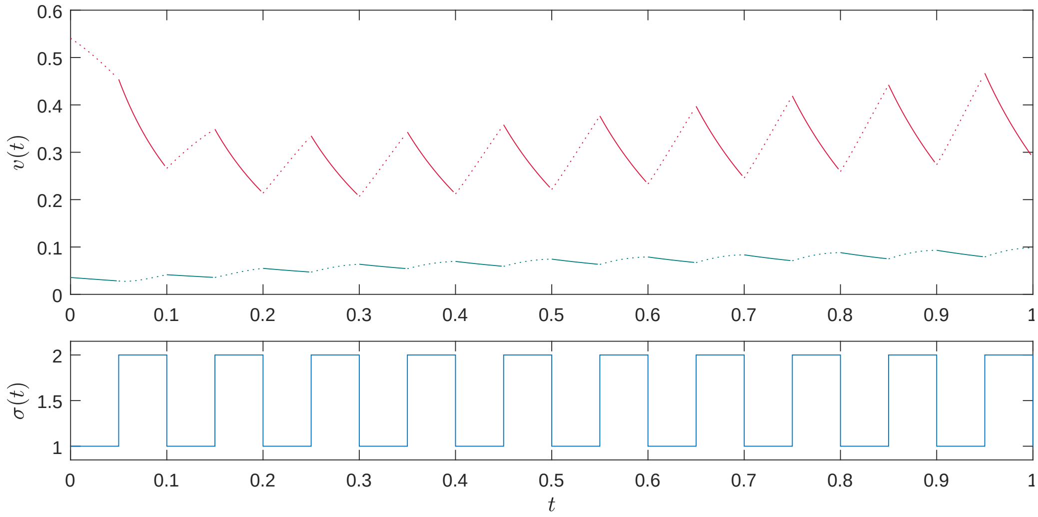


Figure 3.1: Lyapunov function for a destabilizing switching signal.

a suitable switching rule can also be used to stabilize the switched system.

Given its crucial role in the behavior of switched systems, it is important to characterize the switching function $\sigma(\cdot)$, which may be either an arbitrary time-dependent function, or a control variable to be designed. In the first case, the central problem is determining conditions to assure stability for some unknown switching signal $\sigma(t)\colon \mathbb{R}_+ \to \mathbb{K}$, such as a disturbance, an assigned external input, or which may model the effects of a component failure. On the other hand, the second case concerns the design of a switching function $\sigma(\cdot) \in \mathbb{K}$, which can be state, output or input dependent, or a combination thereof, in order to guarantee stability of the switched system. The survey [13] reviews stability conditions for a variety of switched systems that have been introduced in the literature throughout the past decades.

The design of a switching function $\sigma(\cdot)$ as a control variable attracts much interest, as the choice of an appropriate switching rule may assure stability even in the case where all subsystems are unstable, furthermore, whenever all subsystems are stable, it may improve the performance of the overall system when compared to that of each isolated subsystem, in this case the switching function is said to be strictly consistent [25]. This scenario also has a wide scope of applications, for instance, the automatic transmission of an automobile, the problem of temperature regulation by a thermostat, and several applications on power electronic systems. Given that such systems are intrinsically switched, it makes sense to model these in a switched system framework. The following results in the literature, for example, deal with the design of a stabilizing switching rule for switched mode DC-DC power converters, more specifically the *buck-boost*, see [14, 17], and the *flyback* converters, see [18].

It is also relevant to consider measures of performance when designing switching functions for switched systems. In this work, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices, will be presented, as introduced in [23, 24, 25] for the linear case, and in [15, 29] for the affine case. These indices generalize the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms introduced previously, as they cannot be employed, since their definition is given in terms of the transfer function of an LTI system. It should be noted that even though each isolated subsystem may possess a frequency domain representation, the switched system does not, given its nonlinear and time-varying characteristics that stem from the influence of the switching function. It will become evident, however, that whenever the switching function remains fixed at a certain subsystem $\sigma(t) = i$, these indices are equivalent to the square of the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms for the i-th subsystem. These performance criteria will be essential when we treat the problems of control and filter design in the next chapters, as we seek to develop techniques that minimize these indices.

3.2 Stability of Switched Linear Systems

Consider the following state space representation of an unforced continuous-time switched linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_\sigma \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, and $\sigma(\cdot) \in \mathbb{K}$, for $\mathbb{K} := \{1, \ldots, N\}$, is the switching rule, a piecewise continuous function, which selects one of the N available subsystems as active, at every instant of time. Notice that the origin is the single equilibrium point of the system.

In this case, the problem consists in determining an adequate switching rule $\sigma(\cdot)$, capable of stabilizing the system, and making the origin $\mathbf{x} = 0$ a globally asymptotically stable equilibrium point. The work of [19] introduces some circumstances which must be satisfied, so that a stabilizing switching rule is guaranteed to exist. These conditions are derived under Lyapunov's direct method, introduced in Section 2.2, by adopting the quadratic Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$, with $\mathbf{P} \succ 0$, and $\mathbf{x}(t)$ expressed as $\mathbf{x}$ for simplicity.

In the context of linear systems, several well established results for stabilizing switching rules exist, some of which are based on different Lyapunov functions. Before introducing the min-type quadratic rule in the next section, with which many of the results in this work are based upon, we investigate the motivation behind its conception by means of an example, based on [19].

Example 3.1.

Consider the switched linear system (3.1), comprised of two unstable subsystems, with matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. Also, consider symmetric indefinite matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$, a symmetric positive definite matrix $\mathbf{P} \succ 0$, and suppose that a given $\boldsymbol{\lambda}_0 = [\mu \;\; 1-\mu]^T$, with $\mu \in (0, 1)$ exists, such that $\mathbf{A}_{\lambda_0} = \mu \mathbf{A}_1 + (1-\mu)\mathbf{A}_2$ is Hurwitz. Let

$$\mathbf{Q}_1 = \mathbf{A}_1^T \mathbf{P} + \mathbf{P}\mathbf{A}_1 \qquad \text{and} \qquad \mathbf{Q}_2 = \mathbf{A}_2^T \mathbf{P} + \mathbf{P}\mathbf{A}_2$$

Notice that, since $\mathbf{A}_{\lambda_0} \in \mathcal{H}$, and approaching the stability analysis by Lyapunov's direct method under the quadratic Lyapunov function, the following LMI

$$\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0$$

is verified. This implies that, for all $\mathbf{x} \in \mathbb{R}^{n_x}$, $\mathbf{x} \neq \mathbf{0}$,

$$\mathbf{x}^T \left(\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0}\right)\mathbf{x} = \mathbf{x}^T \mathbf{Q}_{\lambda_0}\mathbf{x} < 0$$

indicating that $\mathbf{Q}_{\lambda_0} \prec 0$. Expanding the convex combination in λ_0, we have

$$\mu \left(\mathbf{x}^T \mathbf{Q}_1 \mathbf{x}\right) + (1-\mu) \left(\mathbf{x}^T \mathbf{Q}_2 \mathbf{x}\right) < 0$$

This suggests that either

$$\mathbf{x}^T \mathbf{Q}_1 \mathbf{x} < 0 \qquad \text{or} \qquad \mathbf{x}^T \mathbf{Q}_2 \mathbf{x} < 0$$

However, by our initial hypothesis that both subsystems are not Hurwitz, there does not exist a matrix $\mathbf{P}_1 \succ 0$ such that $\mathbf{A}_1^T \mathbf{P}_1 + \mathbf{P}_1 \mathbf{A}_1 \prec 0$, nor does exist $\mathbf{P}_2 \succ 0$ satisfying $\mathbf{A}_2^T \mathbf{P}_2 + \mathbf{P}_2 \mathbf{A}_2 \prec 0$. This, coupled with the fact that $\mathbf{Q}_{\lambda_0} \prec 0$, implies that indeed $\mathbf{Q}_i$, $i \in \{1,2\}$ are indefinite, and thus, the sign of $\mathbf{x}^T \mathbf{Q}_i \mathbf{x}$ are dependent of the value of $\mathbf{x}$. As such, it can be inferred that while neither subsystem is stable for all $\mathbf{x}(t) \in \mathbb{R}^{n_x}$, as the state trajectory progresses in time, $\mathbf{x}^T \mathbf{Q}_1 \mathbf{x}$ and $\mathbf{x}^T \mathbf{Q}_2 \mathbf{x}$ take turns in becoming strictly negative, as a consequence of $\mathbf{Q}_i$ being indefinite, thus always guaranteeing $\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0$ at any given moment in time.

Notice that $\mathbf{x}^T \left(\mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} = \mathbf{x}^T \mathbf{Q}_i \mathbf{x}$ is precisely the time derivative of the quadratic Lyapunov function for the i-th subsystem. This invites the question of what switching function would select the appropriate subsystem so as to guarantee the time derivative of the Lyapunov function be strictly negative for all instant of time. ■

A recurrent condition in the literature, applicable to the results in this chapter, and employed by this example, is that the convex combination of matrices $\mathbf{A}_{\lambda_0}$, for $\lambda_0 \in \Lambda_N$, be Hurwitz, or $\mathbf{A}_{\lambda_0} \in \mathcal{H}$. This requirement is sufficient to assure that a stabilizing switching rule exists, as discussed in [19, 20].

3.2.1 Switching Rules for Switched Linear Systems

Several switching rules, along with their respective conditions for stability, have been proposed for linear switched systems over the past decades, with varying degrees of conservativeness. Switching rules of the form $\sigma(\mathbf{x}(t))$, $\sigma(\mathbf{y}(t))$, and $\sigma(\mathbf{x}(t), \mathbf{w}(t))$ have been introduced, depending whether these measurements are accessible in order to implement the rule. At the moment, we will focus on switching rules dependent on the state, $\sigma(\mathbf{x}(t))$, and based on quadratic stability [15, 19, 20, 21, 22]. Some other results available in the literature will be shown in brief, towards the conclusion of this section.

3.2.1.1 Quadratic Lyapunov Function

The min-type quadratic rule $\sigma(\mathbf{x}(t))\colon \mathbb{R}^{n_x} \to \mathbb{K}$, introduced by [15, 19, 20, 22], is defined as follows for the switched linear system (3.1)

$$\sigma(\mathbf{x}) = \arg\min_{i\in\mathbb{K}} \mathbf{x}^T\mathbf{P}\mathbf{A}_i\mathbf{x} \tag{3.2}$$

where $\mathbf{P} \in \mathbb{R}^{n_x \times n_x}$, with $\mathbf{P} > 0$. This switching rule guarantees global asymptotic stability of the equilibrium point $\mathbf{x} = \mathbf{0}$ for the system (3.1), considering the quadratic Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$, whenever there exists a vector $\lambda_0 \in \Lambda_N$, such that $\mathbf{A}_{\lambda_0}$ is Hurwitz.

It is interesting to observe that this switching rule is equivalent to

$$\sigma(\mathbf{x}) = \arg\min_{i\in\mathbb{K}} \dot{v}_i(\mathbf{x}) \tag{3.3}$$

indeed, if we recognize that $\dot{v}_i(\mathbf{x}) = \mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x}$ for the i-th subsystem, then

$$\begin{aligned} \sigma(\mathbf{x}) &= \arg\min_{i\in\mathbb{K}} \ \mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} \\ &= \arg\min_{i\in\mathbb{K}} \ 2\,\mathbf{x}^T\mathbf{P}\mathbf{A}_i\mathbf{x} \end{aligned} \tag{3.4}$$

which is equivalent to rule (3.2) as stated.

The following theorem, available in [15], gives the conditions under which the min-type quadratic rule stabilizes the switched linear system (3.1).

Theorem 3.1. *Consider the switched linear system* (3.1) *and a vector* $\lambda_0 \in \Lambda_N$. *If there exists a matrix* $\mathbf{P} > 0$, *such that*

$$\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} < 0 \tag{3.5}$$

then the following switching rule

$$\sigma(\mathbf{x}) = \arg\min_{i\in\mathbb{K}} \mathbf{x}^T\mathbf{P}\mathbf{A}_i\mathbf{x} \tag{3.6}$$

makes the equilibrium point $\mathbf{x}_e = \mathbf{0}$ *globally asymptotically stable.*

Proof. The proof is available on [15], but will be reproduced below for convenience. The stabilizing nature of the min-type quadratic rule for the system (3.1) is demonstrated via Lyapunov's direct method, by verifying that it indeed ensures that the time derivative of the quadratic Lyapunov function is strictly negative for any trajectory $\mathbf{x} \neq \mathbf{0}$, as follows

$$\begin{aligned} \dot{v}(\mathbf{x}) &= \mathbf{x}^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\mathbf{x} \\ &= \min_{i\in\mathbb{K}} \mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} && (3.7) \\ &= \min_{\lambda\in\Lambda_N} \mathbf{x}^T\left(\mathbf{A}_\lambda^T\mathbf{P} + \mathbf{P}\mathbf{A}_\lambda\right)\mathbf{x} && (3.8) \\ &\leq \mathbf{x}^T\left(\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0}\right)\mathbf{x} < 0 && (3.9) \end{aligned}$$

where the equality (3.7) comes from applying the switching rule; equality (3.8) comes from the fact that the minimum of a objective function linear in $\boldsymbol{\lambda}$ always occurs at one of the vertices of the convex polytope defined by Λ_N, thus selecting the i-th subsystem is equivalent to setting the i-th element of $\boldsymbol{\lambda}$ to 1, and the rest to 0; inequality (3.9) stems from the fact that since there exists a vector $\boldsymbol{\lambda}_0$, such that a convex combination of the subsystems is Hurwitz, then at any given time, the minimum of the objective function in $\boldsymbol{\lambda}$ will be always less than or equal to it, and finally, $\dot{v}(\mathbf{x}) < 0$ follows from the fact that $\mathbf{A}_{\lambda_0}$ is Hurwitz, and thus, $\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0$. □

As discussed in Example 3.1, this result implies that at any given moment, for all trajectories $\mathbf{x}(t) \neq 0,\ t \geq 0$, at least one of the subsystems is assured to make $\dot{v}(\mathbf{x}) < 0$, that is, $\exists i \in \mathbb{K}$, such that

$$\mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} < 0 \tag{3.10}$$

for some $\mathbf{x} \in \mathbb{R}^{n_x}$. It is also important to note that no stability property is imposed on the individual subsystems, the only requirement is that $\mathbf{A}_{\lambda_0}$ be Hurwitz for some $\boldsymbol{\lambda}_0 \in \Lambda_N$.

This result will be illustrated in the following numerical example, where we consider switching across two unstable subsystems.

Example 3.2.

Consider the switched linear system (3.1) consisting of the following two unstable subsystems

$$\mathbf{A}_1 = \begin{bmatrix} -10 & 3 \\ 8 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix}$$

Notice that the equilibrium points of both subsystems are unstable saddles, as illustrated in the phase portrait in Figure 3.2. and that at $\boldsymbol{\lambda}_0 = [0.5\ \ 0.5]^T$, a Hurwitz convex combination $\mathbf{A}_{\lambda_0}$ is verified.

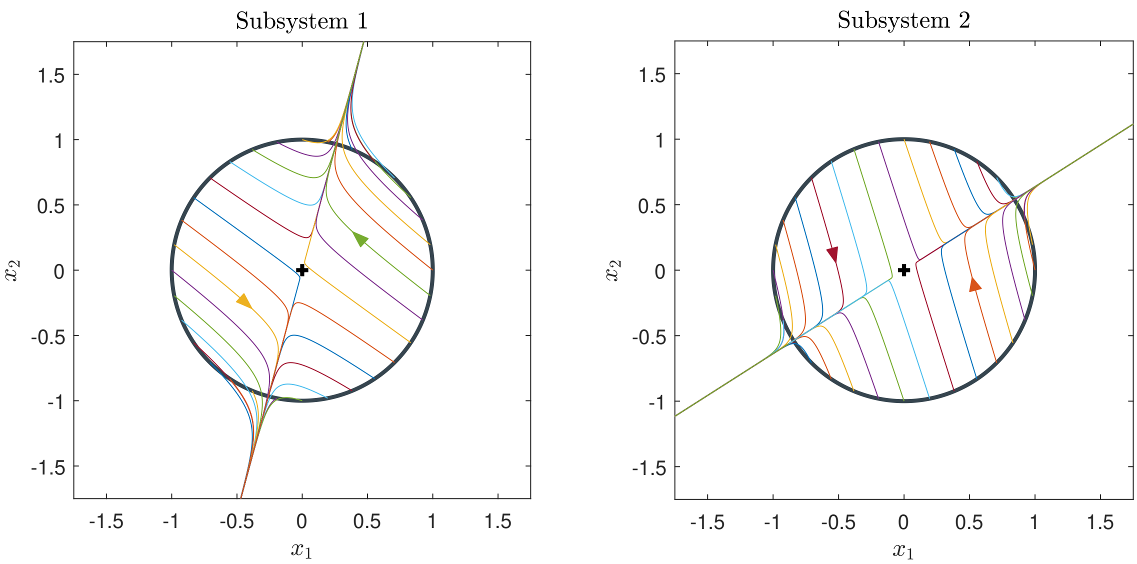


Figure 3.2: Phase portrait for each unforced linear subsystem.

Figure 3.2 displays the trajectories of each isolated subsystem from initial conditions $\mathbf{x}_0$ around a unit circle '—' centered at the origin '+', that is, $\mathbf{x}_0 = [\cos(\theta) \;\; \sin(\theta)]^T$, $\theta \in [0, 2\pi]$. In order to implement the switching rule (3.6) of Theorem 3.1, first, a matrix $\mathbf{P} \succ 0$ is calculated satisfying restriction (3.5) of Theorem 3.1. For the value of λ_0 given, we have considered

$$\mathbf{P} = \begin{bmatrix} 30.2067 & 62.5002 \\ 62.5002 & 135.1940 \end{bmatrix}$$

It can be verified that the switching rule is able to stabilize the switched system, by inspecting the phase portrait and the state trajectories in Figures 3.3 and 3.4[1] which asymptotically converge to the origin, as desired.

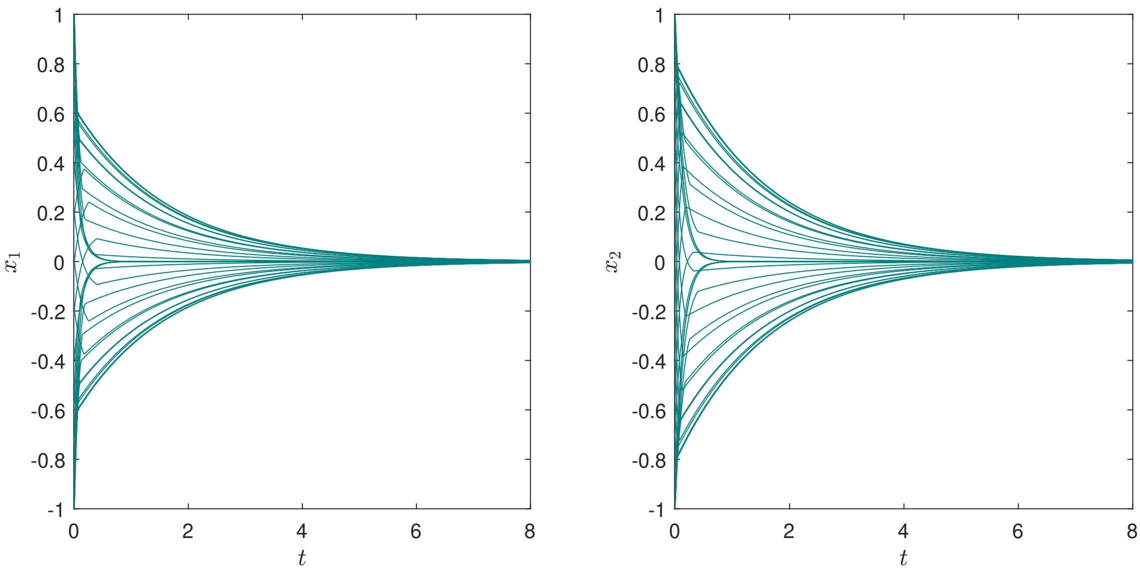


Figure 3.3: Trajectories in time of each state for the switched linear system.

Also, in Figure 3.4 the boundaries in which switching events occur, referred to as a switching surface, are indicated by the line '—'. This situation arises whenever $\mathbf{x}^T\mathbf{P}\mathbf{A}_1\mathbf{x} = \mathbf{x}^T\mathbf{P}\mathbf{A}_2\mathbf{x}$ which, in this case, produces two straight lines intersecting at the origin. Notice that when at the switching surface, the switching rule may select another subsystem as active, transitioning from one dynamical behavior to another, or the occurrence of sliding modes may ensue. In this example, this phenomenon is distinctly visible in Figure 3.4, where the two subsystems switch at an arbitrarily high frequency, resulting in a behavior distinct from that of each isolated subsystem, and causing the state trajectory to evolve along the switching surface towards the equilibrium point, as indicated by the arrows '▸'. ■

[1]Numerical simulations of switched systems were carried out using the SWSYSToolbox for MATLAB, developed by the author. The toolbox is available at `https://github.com/gkolotelo/SWSYSToolbox`, along with the detailed documentation.

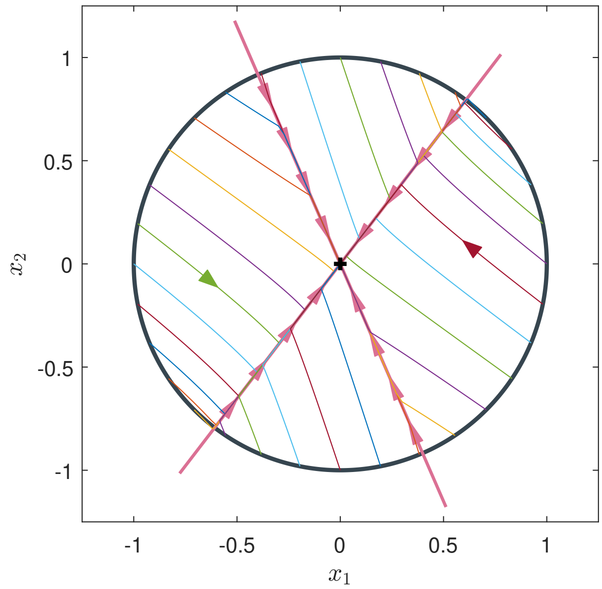


Figure 3.4: Phase portrait for the switched linear system using the min-type quadratic rule.

3.2.1.2 Min-Type Lyapunov Function

As alluded to earlier, less conservative results when compared to those based on quadratic Lyapunov functions have been obtained, such as those derived from multiple Lyapunov functions, introduced by [42, 2], and from the min-type piecewise quadratic Lyapunov function, as shown in [43, 44]. The latter two adopt the following Lyapunov function

$$v(\mathbf{x}) = \min_{i \in \mathbb{K}} \mathbf{x}^T \mathbf{P}_i \mathbf{x} \tag{3.11}$$

with matrices $\mathbf{P}_i \succ 0,\ i \in \mathbb{K}$, the solutions of the well-known Lyapunov-Metzler inequalities

$$\mathbf{A}_i^T \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i + \sum_{j=1}^{N} \pi_{ji} \mathbf{P}_j + \mathbf{E}_i^T \mathbf{E}_i \prec 0,\ i \in \mathbb{K} \tag{3.12}$$

where $\mathbf{\Pi} = \left\{\pi_{ji}\right\}$ is a subclass of Metzler matrices satisfying the following additional property

$$\sum_{j=1}^{N} \pi_{ji} = 0,\ i \in \{1, \ldots, N\} \tag{3.13}$$

Under these conditions, the following switching rule

$$\sigma(\mathbf{x}) = \arg \min_{i \in \mathbb{K}} \mathbf{x}^T \mathbf{P}_i \mathbf{x} \tag{3.14}$$

guarantees global asymptotic stability of the origin. An important remark, as discussed in [45], is that the inequalities (3.12) are less conservative than assuring the existence of a Hurwitz convex combination of subsystem matrices as required in Theorem 3.1. Note that the conditions (3.12) are non-convex due to the product of matrix variables, however, for a small number of subsystems, they can be solved by means of one-dimensional searches with respect to the elements of matrix $\mathbf{\Pi}$ coupled with a set of LMIs. For an arbitrary number of subsystems, alternative conditions, easier to solve but more conservative, are available in [43].

3.3 Stability of Switched Affine Systems

In this section, we introduce the fundamental concepts of switched affine systems that will be extensively used throughout the remainder of this work. Furthermore, we engage in discussions about the unique characteristics of these types of systems, and how these features are useful for modeling many practical applications.

Consider the following state space representation of an unforced continuous-time switched affine system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_\sigma \mathbf{x}(t) + \mathbf{b}_\sigma, \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{z}(t) &= \mathbf{E}_\sigma \mathbf{x}(t) \end{aligned} \tag{3.15}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{b}_\sigma$ is the affine term, $\sigma(\cdot) \in \mathbb{K}$ is the switching rule, and $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance output, which will allow for the definition of performance indices for these systems.

Notice that whenever the affine terms $\mathbf{b}_i = \mathbf{0}$ for all $i \in \mathbb{K}$, the system (3.15) reduces to a continuous-time switched linear system, whose equilibrium point is the origin. However, for the more general case of $\mathbf{b}_i \neq \mathbf{0}$, system (3.15) exhibits several distinct equilibrium points, constituting a subset of the state space. This imposes greater difficulty in the study of the stability properties of switched affine systems, as will soon become evident.

Definition 1. *The set of all equilibrium points of the system* (3.15) *is given by*

$$\mathbf{X}_e = \left\{ \mathbf{x}_e \in \mathbb{R}^{n_x} \ : \ \mathbf{x}_e = -\mathbf{A}_\lambda^{-1} \mathbf{b}_\lambda, \quad \lambda \in \Lambda_N \right\} \tag{3.16}$$

With no loss of generality, system (3.15) can be shifted so as to move the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ to the origin by defining the new state vector $\boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{x}_e$, resulting in the equivalent system

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \\ \mathbf{z}_e(t) &= \mathbf{E}_\sigma \boldsymbol{\xi}(t) \end{aligned} \tag{3.17}$$

where $\boldsymbol{\ell}_i = \mathbf{A}_i \mathbf{x}_e + \mathbf{b}_i$, for all $i \in \mathbb{K}$, are the new affine terms, and $\mathbf{z}_e(t) = \mathbf{z}(t) - \mathbf{E}_\sigma \mathbf{x}_e$ is the shifted output. Notice that for global asymptotic stability, $\boldsymbol{\xi}(t) \to 0$ as $t \to \infty$, and in this condition, $\mathbf{x}(t) \to \mathbf{x}_e$ for system (3.15). Furthermore, observe that whenever $\mathbf{x}_e \in \mathbf{X}_e$, with its associated $\lambda_0 \in \Lambda_N$, we have $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. This choice of state variables will simplify our further developments.

3.3.1 Switching Rules for Switched Affine Systems

On the realm of switched affine systems, fewer switching rules along with suitable Lyapunov functions have been introduced, compared to switched linear systems. The authors [26, 27, 28] present state dependent switching rules, whereas [16, 30] introduce an output dependent switching function. First we present the min-type switching rule, originally devised in [26], along with the conditions which guarantee global asymptotic stability of an equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$. We then introduce the concept of guaranteed cost for switched systems, along with an appropriate switching rule, presented in [14, 15], and capable of assuring this metric, important for the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices towards the end of this chapter.

3.3.1.1 Min-Type Quadratic Rule

The following theorem introduces the results of [15, 26], presenting conditions that assure global asymptotic stability of a desired equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$.

Theorem 3.2. *Consider the switched affine system (3.17), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} > 0$, such that*

$$\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0 \tag{3.18}$$

then the following switching rule

$$\sigma(\xi) = \arg \min_{i \in \mathbb{K}} \; \xi^T \mathbf{P}\mathbf{A}_i \xi + \xi^T \mathbf{P} \boldsymbol{\ell}_i \tag{3.19}$$

makes the equilibrium point $\mathbf{x}_e$ globally asymptotically stable.

Proof. The proof, available in [26], is presented below for convenience, and explained throughout its unraveling. When adopting the switching strategy (3.19), and realizing that it is equivalent to

$$\sigma(\xi) = \arg \min_{i \in \mathbb{K}} \; \xi^T \left(\mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i \right) \xi + 2\xi^T \mathbf{P} \boldsymbol{\ell}_i \tag{3.20}$$

then the time derivative of the quadratic Lyapunov function $\dot{v}(\xi(t))$, for any state trajectory $\xi \neq \mathbf{0}$, is given by

$$\begin{aligned}
\dot{v}(\xi) &= \dot{\xi}^T \mathbf{P} \xi + \xi^T \mathbf{P} \dot{\xi} \\
&= \xi^T \left(\mathbf{A}_\sigma^T \mathbf{P} + \mathbf{P}\mathbf{A}_\sigma \right) \xi + 2\xi^T \mathbf{P} \boldsymbol{\ell}_\sigma \\
&= \min_{i \in \mathbb{K}} \; \xi^T \left(\mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i \right) \xi + 2\xi^T \mathbf{P} \boldsymbol{\ell}_i \\
&= \min_{\lambda \in \Lambda_N} \; \xi^T \left(\mathbf{A}_\lambda^T \mathbf{P} + \mathbf{P}\mathbf{A}_\lambda \right) \xi + 2\xi^T \mathbf{P} \boldsymbol{\ell}_\lambda \\
&\leq \xi^T \left(\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \right) \xi + 2\xi^T \mathbf{P} \boldsymbol{\ell}_{\lambda_0} \\
&< 0
\end{aligned} \tag{3.21}$$

which unfolds in a similar manner to the proof of the min-type quadratic rule for switched linear systems, presented in Section 3.2.1.1, when recalling the fact that $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$, and that $\mathbf{A}_{\lambda_0}$ is Hurwitz, as ensured by condition (3.18), thus making inequality (3.21) valid. □

Notice that this theorem encompasses Theorem 3.1, since in the event that $\mathbf{b}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, they become equivalent. It is also important to observe that not all equilibrium points $\mathbf{x}_e \in \mathbf{X}_e$ are attainable, but in fact only those for which the vectors $\lambda \in \Lambda_N$ satisfy $\mathbf{A}_\lambda \in \mathcal{H}$. In the case where there exists no stable convex combination of dynamical matrices, however, the system is not stabilizable under Theorem 3.2.

Overall, this result is very attractive for a range of real life applications, since it allows the switched system to operate at a chosen equilibrium point of interest. The works of [14, 15, 17, 18] apply this particular characteristic of switched affine systems to different topologies of DC-DC power converters with much success. A numerical example is presented below to demonstrate the validity of Theorem 3.2 with respect to the min-type quadratic rule for switched affine systems.

3.3.1.2 Numerical Example

The following numerical example illustrates the peculiarities of switched affine systems and how the switching function plays an important role in this class of switched systems.

Example 3.3.

Consider the switched affine system (3.15) comprised of a stable and an unstable subsystem, as follows

$$\mathbf{A}_1 = \begin{bmatrix} 8 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -2 & -9 \\ 5 & -4 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

whose respective equilibrium points are

$$\mathbf{x}_{e_1} = \begin{bmatrix} -0.625 \\ -3.1875 \end{bmatrix}, \quad \mathbf{x}_{e_2} = \begin{bmatrix} 1.0377 \\ 0.5472 \end{bmatrix}$$

For the equilibrium point of interest $\mathbf{x}_e = [0.9158 \;\; 0.8160]^T \in \mathbf{X}_e$, with its vector $\lambda_0 = [0.15 \;\; 0.85]^T$ associated, a Hurwitz convex combination $\mathbf{A}_{\lambda_0} \in \mathcal{H}$ is verified. In Figure 3.5, we can observe the dynamical behavior of each isolated subsystem in the ξ phase plane, where '×' denotes the origin of the shifted system. For subsystem 1, we have considered $\xi_0 = \mathbf{x}_0 - \mathbf{x}_{e_1} - \mathbf{x}_e$ describing points over the line '▬' from $\mathbf{x}_0 = [2 \;\; -2]^T$ to $\mathbf{x}_0 = [-2 \;\; 2]^T$. For subsystem 2 we have considered initial conditions $\xi_0 = \mathbf{x}_0 - \mathbf{x}_e$ with $\mathbf{x}_0 = 3 \times [\cos(\theta) \;\; \sin(\theta)]^T$, $\theta \in [0, 2\pi]$, corresponding to points distributed around the circle '▬'. Notice that the equilibrium point $\mathbf{x}_{e_1}$, marked by '◇', is an unstable node, whereas $\mathbf{x}_{e_2}$, indicated by '○', is a stable focus, also, notice how $\mathbf{x}_e$ differs from the equilibrium points of the isolated subsystems. These different behaviors across subsystems bring about complex dynamical behaviors for the switched system.

Before proceeding, the vectors $\boldsymbol{\ell}_i = \mathbf{A}_i \mathbf{x}_e + \mathbf{b}_i$, for $i \in \{1, 2\}$ are calculated. We have also considered matrix

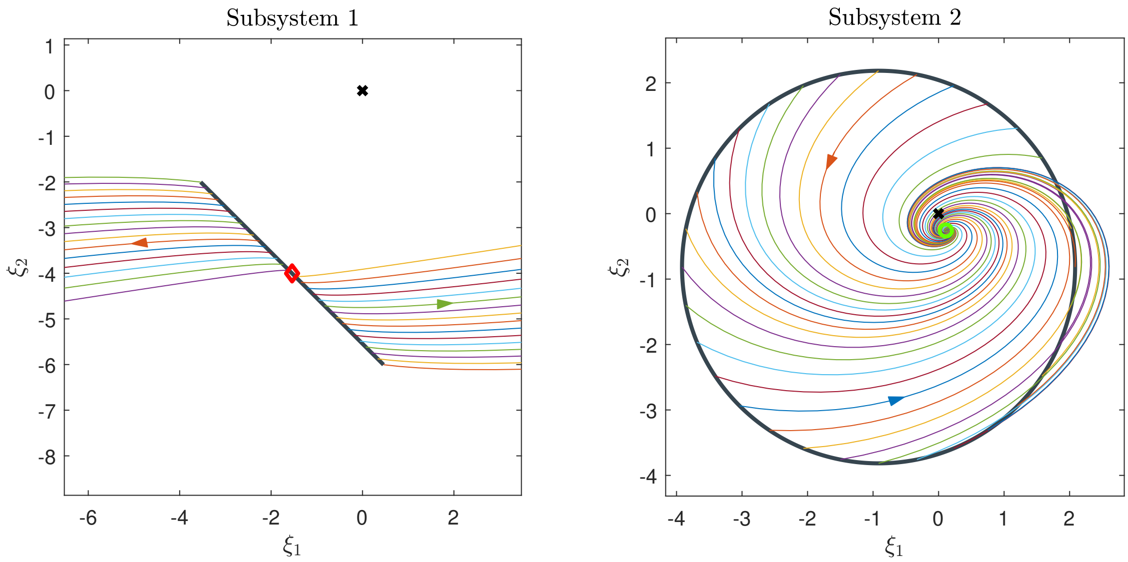


Figure 3.5: Phase portrait for each unforced affine subsystem.

$\mathbf{P} > 0$ as follows

$$\mathbf{P} = \begin{bmatrix} 0.2491 & -0.0650 \\ -0.0650 & 0.4107 \end{bmatrix}$$

which satisfies the LMI restriction (3.18) of Theorem 3.2. Implementing the switching rule (3.19), and allowing initial conditions distributed around the circle '—' with $\xi_0 = [2\cos(\theta)\ \ 2\sin(\theta)]^T + \xi_\bullet$, $\theta \in [0, 2\pi]$, where $\xi_\bullet = [-0.6079\ \ -1.0480]^T$, the switched system is successfully stabilized, as observed in the phase portrait shown in Figure 3.6. Notice that $\xi_\bullet$ is the center of the switching surface, indicated by the line '—', taking place when $\xi^T \mathbf{P}\mathbf{A}_1 \xi + \xi^T \mathbf{P}\boldsymbol{\ell}_1 = \xi^T \mathbf{P}\mathbf{A}_2 \xi + \xi^T \mathbf{P}\boldsymbol{\ell}_2$, forming, in this case, an ellipse.

Notice that when the trajectory is outside this ellipse, it assumes the behavior of subsystem 2, conversely, when inside the ellipse, the trajectory follows the dynamical behavior of subsystem 1. It is interesting to observe that when at the switching surface, the trajectory may exhibit a particular behavior, characteristic of switched system, known as sliding mode. The occurrence of sliding modes is clearly visible in Figure 3.6. This phenomenon, resulting from an arbitrarily fast switching between subsystems, or *chattering*, is sometimes an undesirable condition in real systems, given the increased equipment wear that may ensue. However, it may also be a sought after situation, as it allows for the stability of the overall system. In the case of switched affine systems, sliding modes are crucial, allowing an equilibrium point different than that of each isolated subsystem to be attained, as evidenced by this example.

Figure 3.7 reveals the states of the switched affine system asymptotically reaching the equilibrium point $\xi = \mathbf{0}$, or equivalently, $\mathbf{x} = \mathbf{x}_e$ for the initial conditions considered.

■

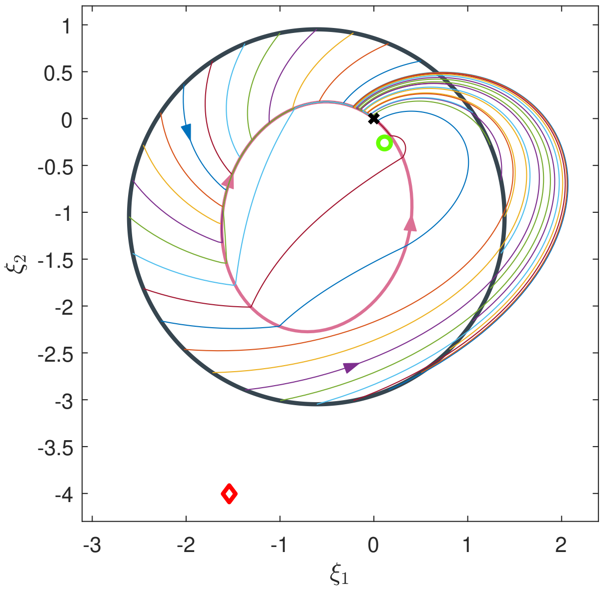


Figure 3.6: Phase portrait for the switched affine system under the min-type quadratic rule.

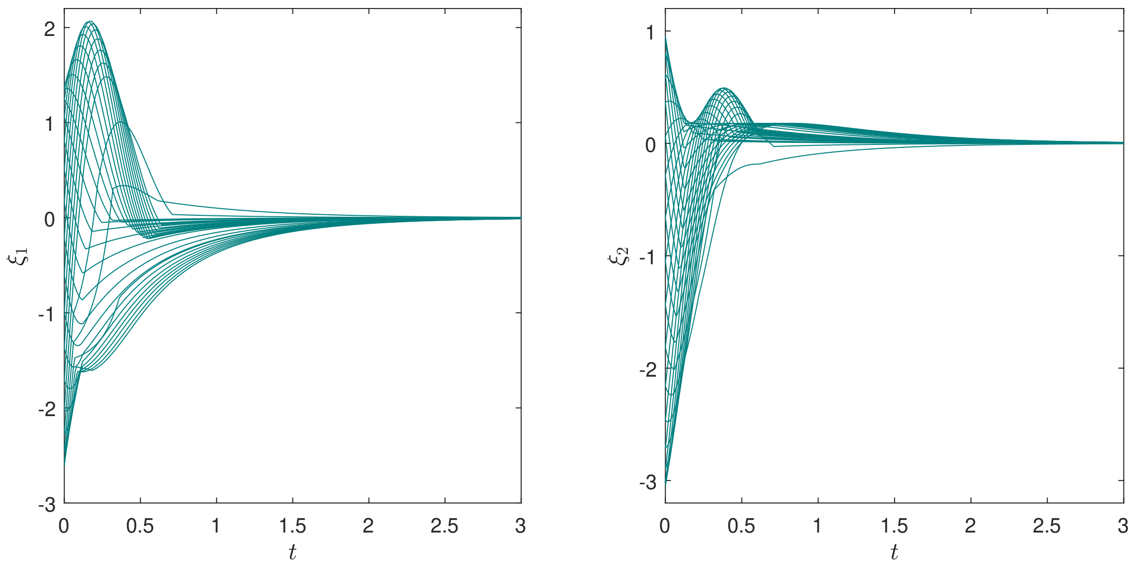


Figure 3.7: Trajectories of each state for the switched affine system under the min-type quadratic rule.

3.3.1.3 Min-Type Quadratic Rule and Guaranteed Cost

A guaranteed cost is an upper bound for a certain performance criteria of a dynamical system. In this case, we consider the cost function as the $\mathcal{L}_2$ norm of the performance output $\|\mathbf{z}_e\|_2$ for the switched affine system (3.17). In this section we demonstrate how the quadratic Lyapunov function can be used to ensure this upper bound for this class of switched systems, as shown in [14, 15]. To this end, the following theorem is borrowed from [15].

Theorem 3.3. *Consider the switched affine system* (3.17)*, and a chosen* $\mathbf{x}_e \in \mathbf{X}_e$ *of interest with its associated* $\lambda_0 \in \Lambda_N$*.*

If there exist a matrix $\mathbf{P} > 0$*, and symmetric indefinite matrices* $\mathbf{Q}_i$*, such that*

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{Q}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{3.22}$$

$$\mathbf{Q}_{\lambda_0} \succeq 0 \tag{3.23}$$

then the following switching rule

$$\sigma(\boldsymbol{\xi}) = \arg\min_{i\in\mathbb{K}} -\boldsymbol{\xi}^T\mathbf{Q}_i\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.24}$$

makes the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$ *globally asymptotically stable, and the guaranteed cost*

$$\|\mathbf{z}_e\|_2^2 < \boldsymbol{\xi}_0^T\mathbf{P}\boldsymbol{\xi}_0 \tag{3.25}$$

holds.

Proof. The proof follows from the definition of the guaranteed cost available in [15]. By implementing the switching strategy (3.24) of Theorem 3.3, the time derivative of the quadratic Lyapunov function $v(\boldsymbol{\xi})$, for any state trajectory $\boldsymbol{\xi} \neq \mathbf{0}$, is given by

$$\begin{aligned}
\dot{v}(\boldsymbol{\xi}) &= \dot{\boldsymbol{\xi}}^T\mathbf{P}\boldsymbol{\xi} + \boldsymbol{\xi}^T\mathbf{P}\dot{\boldsymbol{\xi}} \\
&= \boldsymbol{\xi}^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\sigma + \left(\mathbf{z}_e^T\mathbf{z}_e - \mathbf{z}_e^T\mathbf{z}_e\right) \\
&= \boldsymbol{\xi}^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma + \mathbf{E}_\sigma^T\mathbf{E}_\sigma\right)\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e \\
&< -\boldsymbol{\xi}^T\mathbf{Q}_\sigma\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e \qquad (3.26)\\
&= \min_{i\in\mathbb{K}} -\boldsymbol{\xi}^T\mathbf{Q}_i\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_i - \mathbf{z}_e^T\mathbf{z}_e \\
&= \min_{\lambda\in\Lambda} -\boldsymbol{\xi}^T\mathbf{Q}_\lambda\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\lambda - \mathbf{z}_e^T\mathbf{z}_e \\
&\leq -\boldsymbol{\xi}^T\mathbf{Q}_{\lambda_0}\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\,\boldsymbol{\ell}_{\lambda_0} - \mathbf{z}_e^T\mathbf{z}_e \\
&\leq -\mathbf{z}_e^T\mathbf{z}_e \qquad (3.27)
\end{aligned}$$

where inequality (3.26) comes from the conditions (3.22) of Theorem 3.3, and inequality (3.27) stems from the fact that $\mathbf{Q}_{\lambda_0} \succeq 0$. Thus, global asymptotic stability is assured for the equilibrium point $\mathbf{x}_e$, as $\mathbf{z}_e^T\mathbf{z}_e > 0$. Finally, by integrating both sides of (3.27) we have

$$\begin{aligned}
\int_0^\infty \dot{v}(\boldsymbol{\xi})\,\mathrm{d}t &< -\int_0^\infty \mathbf{z}_e(t)^T\mathbf{z}_e(t)\,\mathrm{d}t \\
\lim_{t\to\infty} v(\boldsymbol{\xi}(t)) - v(\boldsymbol{\xi}_0) &< -\int_0^\infty \mathbf{z}_e(t)^T\mathbf{z}_e(t)\,\mathrm{d}t \\
\int_0^\infty \mathbf{z}_e(t)^T\mathbf{z}_e(t)\,\mathrm{d}t &< v(\boldsymbol{\xi}_0) \\
\|\mathbf{z}_e\|_2^2 &< \boldsymbol{\xi}_0^T\mathbf{P}\boldsymbol{\xi}_0
\end{aligned} \tag{3.28}$$

where $\lim_{t\to\infty} v(\boldsymbol{\xi}(t)) = 0$, since the system is asymptotically stable. This assures the guaranteed cost for the system $\|\mathbf{z}_e\|_2^2 < \xi_0^T \mathbf{P} \xi_0$ whenever $\mathbf{z}_e(t)$ is square-integrable. □

It is worth noticing that, as $\mathbf{Q}_{\lambda_0} \geq 0$, the following can be verified

$$\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} + \sum_{i=0}^{N} \lambda_i \mathbf{E}_i^T \mathbf{E}_i \prec 0 \tag{3.29}$$

which shows that the matrices $\mathbf{E}_i$ directly influence $\mathbf{P}$, and consequently, the guaranteed cost for the system, as should be expected. Furthermore, the above condition requires that $\mathbf{A}_{\lambda_0}$ be Hurwitz. This, however, is not an onerous imposition, since matrices $\mathbf{Q}_i$, $\forall i \in \mathbb{K}$, are indefinite, thus no stability property is required from the subsystem matrices $\mathbf{A}_i$ themselves, for all $i \in \mathbb{K}$.

To implement Theorem 3.3, matrices $\mathbf{P}$ and $\mathbf{Q}_i$, for $i \in \mathbb{K}$, important for the switching rule (3.24), can be calculated numerically by solving the following convex optimization problem, subject to the LMI restrictions (3.22) and (3.23) of Theorem 3.3, that is

$$\begin{aligned} \min \quad & \xi_0^T \mathbf{P} \xi_0 \\ \text{subject to:} \quad & \mathbf{P} \succ 0 \\ & \mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i + \mathbf{Q}_i \prec 0, \ \forall i \in \mathbb{K} \\ & \mathbf{Q}_{\lambda_0} \succeq 0 \end{aligned} \tag{3.30}$$

This result is important for the definition of an upper bound for the $\mathcal{H}_2$ performance index, which will be introduced shortly.

The following corollary proposes a min-type linear switching rule, possible in the event that all subsystems are individually stable.

Corollary 3.1. *Consider the switched affine system (3.17), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\boldsymbol{\lambda}_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} \succ 0$, such that*

$$\mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{3.31}$$

then the following switching rule

$$\sigma(\boldsymbol{\xi}) = \arg \min_{i \in \mathbb{K}} \boldsymbol{\xi}^T \mathbf{P}\mathbf{A}_i \mathbf{x}_e \tag{3.32}$$

makes the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$ globally asymptotically stable, and the guaranteed cost

$$\|\mathbf{z}_e\|_2^2 < \xi_0^T \mathbf{P} \xi_0 \tag{3.33}$$

holds.

Proof. The proof is a direct consequence of Theorem 3.3. By imposing $\mathbf{Q}_i \geq \mathbf{0}$, for all $i \in \mathbb{K}$. □

The condition in the corollary requires that all the subsystems be quadratically stable, that is, the subsystems must be Hurwitz stable and, furthermore, they must admit the same matrix $\mathbf{P} \succ 0$. Although this result is more conservative when compared to Theorem 3.3, it is useful when considering practical applications, as it guarantees global asymptotic stability for any $\mathbf{x}_e \in \mathbf{X}_e$, such that $\mathbf{A}_{\lambda_0} \in \mathcal{H}$, by only calculating the matrix $\mathbf{P}$ once, as opposed to Theorem 3.3, in which $\mathbf{P}$ depends on λ_0. This allows for the equilibrium point to change at execution time by simply altering $\mathbf{x}_e$ on the switching rule (3.32).

3.4 Performance Indices

This section aims to introduce the concept of performance indices for switched systems, as well as proposes suboptimal switching rules that assure an upper bound for these indices. For the $\mathcal{H}_2$ case, the results of Theorem 3.3 will be generalized, and a state dependent switching function is proposed. For the $\mathcal{H}_\infty$ case, two switching rules are introduced, one dependent on the state, and another that additionally depends on the disturbance. These concepts have already been applied in the context of switched linear systems as in [23, 24, 25], as well as in the context of switched affine systems [16, 29].

For this section, the following switched affine system is considered, in its shifted representation, such that the origin is the unique equilibrium point

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \mathbf{H}_\sigma \mathbf{w}(t) + \boldsymbol{\ell}_\sigma, \quad \boldsymbol{\xi}(0) = \mathbf{0} \\ \mathbf{z}_e(t) &= \mathbf{E}_\sigma \boldsymbol{\xi}(t) + \mathbf{G}_\sigma \mathbf{w}(t) \end{aligned} \tag{3.34}$$

where $\boldsymbol{\xi}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{\ell}_\sigma$ is the affine term, $\sigma(\cdot) \in \mathbb{K}$ is the switching rule, $\mathbf{z}_e(t) \in \mathbb{R}^{n_z}$ is the performance output, and $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the external disturbance input. The following performance indices will establish a relationship between this disturbance and the output $\mathbf{z}_e(t)$.

3.4.1 $\mathcal{H}_2$ Performance Index for Switched Systems

As in the case for the $\mathcal{H}_2$ norm of LTI systems, $\mathbf{G}_i = \mathbf{0}$ for all $i \in \mathbb{K}$ must be considered to ensure that the output $\mathbf{z}_e(t)$, associated to external impulsive disturbances, be square-integrable.

The $\mathcal{H}_2$ performance index is defined as

$$J_2(\sigma) = \sum_{k=1}^{n_w} \|\mathbf{z}_{e_k}\|_2^2 \tag{3.35}$$

where $\mathbf{z}_{e_k}(t)$ refers to the output of system (3.34) when subject to an impulsive disturbance applied to the k-th input, in other words, $\mathbf{w}(t) = \delta(t)\boldsymbol{\psi}_k$, such that $\boldsymbol{\psi}_k$, for $k \in \{1, \ldots, n_w\}$ form a standard basis. Considering this, observe that (3.35) can be expressed as

$$J_2(\sigma) = \sum_{k=1}^{n_w} \int_0^\infty \mathbf{z}_{e_k}^T(t) \mathbf{z}_{e_k}(t)\, \mathrm{d}t = \sum_{k=1}^{n_w} \sum_{i=1}^{n_z} \int_0^\infty z_{e_{ik}}(t)^2\, \mathrm{d}t \tag{3.36}$$

It is interesting to observe that minimizing $J_2(\sigma)$ generally leads to a faster convergence to the equilibrium point.

For an LTI system, the $\mathcal{H}_2$ norm, shown in (2.22), can be defined in terms of the frequency response of the system, this procedure, however, cannot be employed for switched systems, as they do not admit a transfer matrix due to the effect of switching rule. Nevertheless, notice that for a fixed switching rule, that is $\sigma(t) = i$, for all $t \geq 0$, the $\mathcal{H}_2$ index is indeed equivalent to the square of the $\mathcal{H}_2$ norm since in this case $z_{e_{ik}}(t)$, for an impulsive disturbance, equates to $h_{ik}(t)$ of (2.22).

To deal with switched systems, the upper bound for the $\mathcal{H}_2$ index is defined by considering the guaranteed cost introduced in Theorem 3.3, which was obtained with $\mathbf{w}(t) = \mathbf{0}$, and an arbitrary initial condition. Notice that the system (3.34) subjected to an impulsive disturbance $\mathbf{w}(t) = \delta(t)\boldsymbol{\psi}_k$ can be cast as system (3.17) when $\boldsymbol{\xi}(0) = \mathbf{H}_{\sigma(0)}\boldsymbol{\psi}_k$ is assigned. This can be verified by integrating (3.34)

$$
\begin{aligned}
\int_0^t \dot{\boldsymbol{\xi}}(t)\,\mathrm{d}t &= \int_0^t \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma\,\mathrm{d}t + \int_0^t \mathbf{H}_\sigma \mathbf{w}(t)\,\mathrm{d}t \\
\boldsymbol{\xi}(t) - \boldsymbol{\xi}(0) &= \int_0^t \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma\,\mathrm{d}t + \mathbf{H}_{\sigma(0)}\boldsymbol{\psi}_k \\
\boldsymbol{\xi}(t) - \mathbf{H}_{\sigma(0)}\boldsymbol{\psi}_k &= \int_0^t \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma\,\mathrm{d}t
\end{aligned}
\tag{3.37}
$$

Hence, the upper bound for the $\mathcal{H}_2$ index can be calculated by using the familiar result (3.25) from Theorem 3.3

$$
\begin{aligned}
J_2(\sigma) &= \sum_{k=1}^{n_w} \int_0^\infty \mathbf{z}_{e_k}^T(t)\mathbf{z}_{e_k}(t)\,\mathrm{d}t \\
&< \sum_{k=1}^{n_w} \boldsymbol{\xi}^T(0)\mathbf{P}\boldsymbol{\xi}(0) \\
&= \sum_{k=1}^{n_w} \boldsymbol{\psi}_k^T \mathbf{H}_j^T \mathbf{P}\mathbf{H}_j \boldsymbol{\psi}_k \\
&= \mathrm{tr}\left(\mathbf{H}_j^T \mathbf{P}\mathbf{H}_j\right)
\end{aligned}
\tag{3.38}
$$

where $\mathbf{P}$ satisfies theorem 3.3, and $\sigma(0) = j$ is the initial value of the switching rule, chosen appropriately. Two choices for this value may be of interest, firstly, the choice of $j \in \mathbb{K}$ such that the upper bound J_2 is minimized, providing the smallest guaranteed $\mathcal{H}_2$ performance index; and secondly, a choice of j that maximizes the upper bound J_2, thus making the $\mathcal{H}_2$ index robust with respect to the initial subsystem $\sigma(0) \in \mathbb{K}$. More information of this topic can be found in the reference [23].

Finally, it is also worth mentioning that, although this definition has been established for the shifted system (3.17) considering the performance output $\mathbf{z}_e(t)$, this result equally guarantees an upper bound for the system (3.15) with the performance output $\mathbf{z}(t)$, since these definitions are related by $\mathbf{z}_e(t) = \mathbf{z}(t) - \mathbf{E}_\sigma \mathbf{x}_e$.

With this, the following theorem can be stated

Theorem 3.4. *Consider the switched affine system* (3.34), *with* $\mathbf{G}_i = 0$ *for all* $i \in \mathbb{K}$, *and a chosen* $\mathbf{x}_e \in \mathbf{X}_e$ *of interest* *with its associated $\lambda_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} > 0$, and symmetric indefinite matrices $\mathbf{Q}_i$, such that*

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{Q}_i < 0, \quad \forall i \in \mathbb{K} \tag{3.39}$$

$$\mathbf{Q}_{\lambda_0} \geq 0 \tag{3.40}$$

then the following switching rule

$$\sigma(\boldsymbol{\xi}) = \arg\min_{i\in\mathbb{K}} -\boldsymbol{\xi}^T\mathbf{Q}_i\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.41}$$

makes the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$ globally asymptotically stable, and assures the $\mathcal{H}_2$ guaranteed cost

$$J_2(\sigma) < \mathrm{tr}\left(\mathbf{H}_j^T\mathbf{P}\mathbf{H}_j\right) \tag{3.42}$$

with $\sigma(0) = j$.

Proof. The proof follows from Theorem 3.3, taking into account the relations (3.37) and (3.38). □

The following convex optimization problem allows for the numerical calculation of matrices $\mathbf{P}$ and $\mathbf{Q}_i$, for $i \in \mathbb{K}$ required by the switching rule (3.41) of Theorem 3.4.

$$\begin{aligned} \min \quad & \mathrm{tr}\left(\mathbf{H}_j^T\mathbf{P}\mathbf{H}_j\right) \\ \text{subject to:} \quad & \mathbf{P} > 0 \\ & \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{Q}_i < 0,\ \forall i \in \mathbb{K} \\ & \mathbf{Q}_{\lambda_0} \geq 0 \end{aligned} \tag{3.43}$$

with $j \in \mathbb{K}$ chosen appropriately. In this manner, the upper bound for the $\mathcal{H}_2$ performance index is given by $J_2(\sigma) < \mathrm{tr}\left(\mathbf{H}_j^T\mathbf{P}\mathbf{H}_j\right)$.

3.4.1.1 Numerical Example

A numerical example is provided to compare the $\mathcal{H}_2$ performance index guaranteed by Theorem 3.4 with the $\mathcal{L}_2$ norm of the performance output obtained via numerical simulation.

Example 3.4.

Consider the switched affine system of Example 3.3, with

$$\mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}, \quad \mathbf{H}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

A Hurwitz convex combination $\mathbf{A}_{\lambda_0}$ occurs at $\lambda_0 = [0.2 \ \ 0.8]^T$, with the associated equilibrium point $\mathbf{x}_e = [0.8492 \ \ 0.9167]^T$. In order to implement the switching rule (3.41) of Theorem 3.4, the vectors $\boldsymbol{\ell}_i = \mathbf{A}_i\mathbf{x}_e + \mathbf{b}_i$ have been first calculated, and the matrices $\mathbf{P}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ computed by solving the convex optimization problem (3.43), resulting in the following matrices

$$\mathbf{P} = \begin{bmatrix} 0.3290 & -0.1190 \\ -0.1190 & 0.4847 \end{bmatrix}, \quad \mathbf{Q}_1 = \begin{bmatrix} -6.0266 & 0.7058 \\ 0.7058 & -2.9389 \end{bmatrix}, \quad \mathbf{Q}_2 = \begin{bmatrix} 1.5066 & -0.1765 \\ -0.1765 & 0.7347 \end{bmatrix}$$

associated to the guaranteed cost $J_2(\sigma) < 0.3290$ for $\sigma(0) = 1$. For comparison, by choosing $\sigma(0) = 2$, the guaranteed cost for the switched system would be $J_2(\sigma) < 1.0518$, and would be robust against the choice of $\sigma(0)$. The trajectories in time for each state can be seen in Figure 3.8 with the initial condition $\xi(0) = \mathbf{H}_1\psi_1 = \mathbf{H}_1$, since only a single input channel exists as discussed in Section 3.4.1. The figure indicates the system state trajectories asymptotically converging to $\xi = \mathbf{0}$.

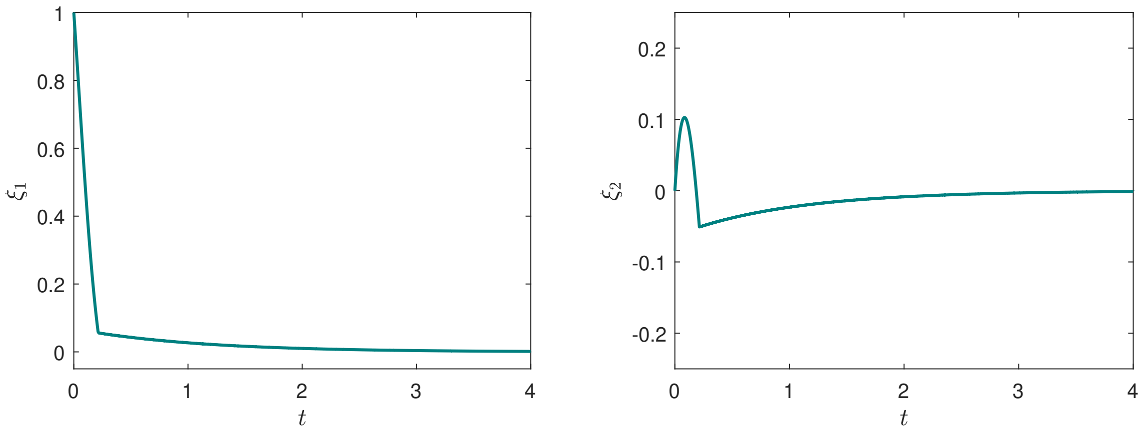


Figure 3.8: Trajectories of each state for the switched affine system under Theorem 3.4.

Finally, by numerical integration of the output signal $\mathbf{z}_e(t)^T\mathbf{z}_e(t)$ from $t = 0$ to $t \to \infty$, the actual $\mathcal{H}_2$ cost obtained from the suboptimal switching function implementation (3.41) was calculated as being $J_2(\sigma) = 0.0705 < 0.3290$, within the cost guaranteed by Theorem 3.4, as expected. ■

3.4.2 $\mathcal{H}_\infty$ Performance Index for Switched Systems

In a manner analogous to the definition of the $\mathcal{H}_\infty$ norm (2.30) established for LTI systems, the $\mathcal{H}_\infty$ performance index for switched systems is defined as

$$J_\infty(\sigma) := \sup_{0 \neq \mathbf{w} \in \mathcal{L}_2} \frac{\|\mathbf{z}_e(t)\|_2^2}{\|\mathbf{w}(t)\|_2^2} < \rho, \quad \rho > 0 \tag{3.44}$$

with the scalar ρ being an upper bound for this index.

In a similar manner to the procedure used in the proof of Theorem 3.3, the upper bound for the $\mathcal{H}_\infty$ index can be inferred from the time derivative of the quadratic Lyapunov function $v(\xi)$, for state trajectories

$\xi \neq \mathbf{0}$, as follows

$$\begin{aligned}
\dot{v}(\xi) &= \dot{\xi}^T \mathbf{P}\xi + \xi^T \mathbf{P}\dot{\xi} \\
&= \xi^T \left(\mathbf{A}_\sigma^T \mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\xi + \mathbf{w}^T \mathbf{H}_\sigma^T \mathbf{P}\xi + \xi^T \mathbf{P}\mathbf{H}_\sigma \mathbf{w} + 2\xi^T \mathbf{P}\boldsymbol{\ell}_\sigma + \left(\mathbf{z}_e^T \mathbf{z}_e - \mathbf{z}_e^T \mathbf{z}_e\right) + \rho\left(\mathbf{w}^T\mathbf{w} - \mathbf{w}^T\mathbf{w}\right) \\
&= \xi^T \left(\mathbf{A}_\sigma^T \mathbf{P} + \mathbf{P}\mathbf{A}_\sigma + \mathbf{E}_\sigma^T \mathbf{E}_\sigma\right)\xi + \mathbf{w}^T \left(\mathbf{H}_\sigma^T \mathbf{P} + \mathbf{G}_\sigma^T \mathbf{E}_\sigma\right)\xi + \xi^T \left(\mathbf{P}\mathbf{H}_\sigma + \mathbf{E}_\sigma^T \mathbf{G}_\sigma\right)\mathbf{w} + \\
&\qquad \mathbf{w}^T \left(\mathbf{G}_\sigma^T \mathbf{G}_\sigma - \rho \mathbf{I}\right)\mathbf{w} + 2\xi^T \mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&= \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \begin{bmatrix} \mathbf{A}_\sigma^T \mathbf{P} + \mathbf{P}\mathbf{A}_\sigma + \mathbf{E}_\sigma^T \mathbf{E}_\sigma & \mathbf{P}\mathbf{H}_\sigma + \mathbf{E}_\sigma^T \mathbf{G}_\sigma \\ \mathbf{H}_\sigma^T \mathbf{P} + \mathbf{G}_\sigma^T \mathbf{E}_\sigma & \mathbf{G}_\sigma^T \mathbf{G}_\sigma - \rho \mathbf{I} \end{bmatrix} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w}
\end{aligned} \tag{3.45}$$

Two switching strategies are introduced, one relying on state information and the other also dependent on the disturbance measurement, which along with their respective conditions for stability, will provide different $\mathcal{H}_\infty$ guaranteed costs that will be shown to have different degrees of conservativeness. The results of this section, concerning the $\mathcal{H}_\infty$ guaranteed cost, are also available in [29].

3.4.2.1 Rule with state and input information

The following theorem states conditions for the control design problem of a stabilizing switching function dependent on the system state $\xi(t)$ and also on the external disturbance $\mathbf{w}(t)$, providing an upper bound for the $\mathcal{H}_\infty$ performance index, as presented in [29].

Theorem 3.5. *Consider the switched affine system (3.34), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} > 0$, and a scalar $\rho > 0$, such that*

$$\sum_{i \in \mathbb{K}} \lambda_{0_i} \mathcal{L}_i(\rho, \mathbf{P}) \prec 0 \tag{3.46}$$

with

$$\mathcal{L}_i(\rho, \mathbf{P}) = \begin{bmatrix} \mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i & \bullet \\ \mathbf{H}_i^T \mathbf{P} + \mathbf{G}_i^T \mathbf{E}_i & \mathbf{G}_i^T \mathbf{G}_i - \rho \mathbf{I} \end{bmatrix} \tag{3.47}$$

then the following switching rule

$$\sigma(\xi, \mathbf{w}) = \arg \min_{i \in \mathbb{K}} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P}\boldsymbol{\ell}_i \tag{3.48}$$

makes the equilibrium point $\mathbf{x}_e$ globally asymptotically stable, and guarantees the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma) < \rho$.

Proof. The proof is available in [29], but is demonstrated here for convenience. By adopting the switching strategy in Theorem 3.5, the time derivative of the quadratic Lyapunov function $\dot{v}(\xi(t))$, considered for an arbitrary state trajectory $\xi \neq \mathbf{0}$, is calculated from (3.45) as follows

$$\begin{aligned}
\dot{v}(\xi) &= \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_\sigma(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_\sigma - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&= \min_{i \in \mathbb{K}} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_i - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&= \min_{\lambda \in \Lambda} \sum_{i \in \mathbb{K}} \lambda_i \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_\lambda - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&\leq \sum_{i \in \mathbb{K}} \lambda_{0_i} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_{\lambda_0} - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&< -\mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w}
\end{aligned} \tag{3.49}$$

where inequality (3.49) comes from condition (3.46), and by recalling that $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. Notice that, for $\mathbf{w}(t) = \mathbf{0}$, we have $\dot{v}(\xi) < 0$, and as such, the equilibrium point $\xi = \mathbf{0}$ is globally asymptotically stable. This allows for the calculation of the upper bound ρ given by

$$\begin{aligned}
\dot{v}(\xi) &< -\mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
\int_0^\infty \dot{v}(\xi)\, \mathrm{d}t &< -\int_0^\infty \mathbf{z}_e^T \mathbf{z}_e\, \mathrm{d}t + \rho \int_0^\infty \mathbf{w}^T \mathbf{w}\, \mathrm{d}t \\
\lim_{t \to \infty} v(\xi(t)) - v(\xi(0)) &< -\|\mathbf{z}_e\|_2^2 + \rho \|\mathbf{w}\|_2^2 \\
\|\mathbf{z}_e\|_2^2 &< \rho \|\mathbf{w}\|_2^2
\end{aligned} \tag{3.50}$$

where $\lim_{t \to \infty} v(\xi(t)) = 0$, for an asymptotically stable system, and $v(\xi(0)) = 0$, since $\xi(0) = \mathbf{0}$. For any disturbance $\mathbf{w}(t) \in \mathcal{L}_2$, $\mathbf{w}(t) \neq 0$, ρ gives the upper bound for the $\mathcal{H}_\infty$ performance index

$$J_\infty(\sigma) = \sup_{0 \neq \mathbf{w} \in \mathcal{L}_2} \frac{\|\mathbf{z}_e\|_2^2}{\|\mathbf{w}\|_2^2} < \rho \tag{3.51}$$

as desired. □

Theorem 3.5 may be implemented by solving the following convex optimization problem, providing matrix $\mathbf{P}$, necessary to the switching rule (3.48)

$$\begin{aligned}
\min \quad & \rho \\
\text{subject to:} \quad & \mathbf{P} \succ 0,\ \rho > 0 \\
& \sum_{i \in \mathbb{K}} \lambda_{0_i} \mathcal{L}_i(\rho, \mathbf{P}) \prec 0
\end{aligned} \tag{3.52}$$

3.4.2.2 Rule with state information

A second switching strategy dependent only on state information $\sigma(\xi)$ is of great relevance, given that the disturbance $\mathbf{w}(t)$ is most often unavailable for measurement. To accomplish this, a disturbance $\mathbf{w}(t)^\star$ is considered, such that it maximizes the product

$$\sup_{\mathbf{w}\in\mathcal{L}_2} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho,\mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} \tag{3.53}$$

By defining the matrix product $f(\xi,\mathbf{w})$ as

$$f(\xi,\mathbf{w}) = \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho,\mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} = \xi^T \left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i\right)\xi + 2\xi^T\left(\mathbf{P}\mathbf{H}_i + \mathbf{E}_i^T\mathbf{G}_i\right)\mathbf{w} + \mathbf{w}^T\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)\mathbf{w} \tag{3.54}$$

taking the partial derivative of $f(\xi,\mathbf{w})$ with respect to $\mathbf{w}(t)$, and setting it to zero

$$\frac{\partial}{\partial\mathbf{w}} f(\xi,\mathbf{w}) = 2\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\xi + 2\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)\mathbf{w} = 0 \tag{3.55}$$

the disturbance input $\mathbf{w}(t)^\star$ can be calculated

$$\mathbf{w}(t)^\star = -\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\xi(t) \tag{3.56}$$

Notice that $\mathbf{w}(t)^\star$ indeed maximizes $f(\xi,\mathbf{w})$, evidenced by evaluating the second derivative of $f(\xi,\mathbf{w})$ with respect to $\mathbf{w}(t)$

$$\frac{\partial^2}{\partial\mathbf{w}^2} f(\xi,\mathbf{w}) = 2\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right) \tag{3.57}$$

whenever $(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I})$ is negative definite.

Substituting $\mathbf{w}(t)^\star = -\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\xi(t)$ for the disturbance $\mathbf{w}(t)$ in $f(\xi,\mathbf{w})$, thus eliminating its dependency on the disturbance, we obtain

$$f(\xi) = \xi^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i - \left(\mathbf{P}\mathbf{H}_i + \mathbf{E}_i^T\mathbf{G}_i\right)\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\right)\xi \tag{3.58}$$

By defining the new matrix term $\mathcal{N}_i(\rho,\mathbf{P})$ as

$$\mathcal{N}_i(\rho,\mathbf{P}) = \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i - \left(\mathbf{P}\mathbf{H}_i + \mathbf{E}_i^T\mathbf{G}_i\right)\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right) \tag{3.59}$$

and introducing symmetric indefinite matrices $\mathbf{Q}_i$, for $i \in \mathbb{K}$, then the following inequalities are satisfied

$$\mathcal{N}_i(\rho,\mathbf{P}) + \mathbf{Q}_i \prec 0 \tag{3.60}$$

$$\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right) \prec 0 \tag{3.61}$$

whenever the following condition is verified

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{Q}_i & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{3.62}$$

Indeed, by successively applying Schur complement with respect to $-\mathbf{I}$ and $(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I})$, we regain the previous inequalities. It is based on this condition, that the theorem presented below guarantees global asymptotic stability, while assuring an upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma) < \rho$.

Theorem 3.6. *Consider the switched affine system (3.34), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\boldsymbol{\lambda}_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} > 0$, symmetric indefinite matrices $\mathbf{Q}_i$, and a scalar ρ such that*

$$\mathbf{Q}_{\lambda_0} \geq 0 \tag{3.63}$$

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{Q}_i & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{3.64}$$

then the switching rule

$$\sigma(\xi) = \arg\min_{i\in\mathbb{K}} \; -\xi^T\mathbf{Q}_i\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.65}$$

makes the equilibrium point $\xi = \mathbf{0}$ globally asymptotically stable and guarantees the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma) < \rho$.

Proof. The complete proof can be found in [29], but is discussed here for use in the next chapters. Adopting the switching strategy defined above, the time derivative of the quadratic Lyapunov function $v(\xi)$, for any trajectory $\xi \neq \mathbf{0}$, is calculated from (3.45) as

$$\begin{aligned}
\dot{v}(\xi) &= \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_\sigma(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq \sup_{\mathbf{w}\in\mathcal{L}_2} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_\sigma(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.66) \\
&= \xi^T\mathcal{N}_\sigma(\rho, \mathbf{P})\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.67) \\
&< -\xi^T\mathbf{Q}_\sigma\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.68) \\
&= \min_{i\in\mathbb{K}} \; -\xi^T\mathbf{Q}_i\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq \min_{\lambda\in\Lambda} \; -\xi^T\mathbf{Q}_\lambda\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\lambda - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq \; -\xi^T\mathbf{Q}_{\lambda_0}\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_{\lambda_0} - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq -\mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.69)
\end{aligned}$$

where equality (3.67) comes from the fact that from definition (3.54) the supremum indicated in (3.66) provides (3.58) with $\mathcal{N}_i(\rho, \mathbf{P})$ defined in (3.59); inequality (3.68) stems from (3.64) which is equivalent to (3.60); and inequality (3.69) arises from condition (3.63), and by realizing that $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. Once again, we have that, for $\mathbf{w}(t) = \mathbf{0}$, $\dot{v}(\xi) < 0$, and as a consequence, the equilibrium point $\xi = \mathbf{0}$ is globally asymptotically stable, and the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma) < \rho$ is guaranteed, as previously demonstrated in the proof of Theorem 3.5. □

It should be noted that Theorems 3.5 and 3.6 do not impose that $\mathbf{A}_i$ be Hurwitz, for all $i \in \mathbb{K}$, because of the presence of matrices $\mathbf{Q}_i$, $i \in \mathbb{K}$ as discussed in the previous sections. However, conditions (3.46) and (3.64) do require that $\mathbf{A}_\lambda \in \mathcal{H}$, as has been recurrent thus far. Furthermore, an important aspect of Theorem 3.6 is that the conditions are based on the fact that $\mathbf{Q}_\lambda \geq 0$, and as such, $\sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(\rho, \mathbf{P}) < 0$, but due to the product between indexed matrices, specifically $\mathbf{E}_i^T \mathbf{E}_i$, $\mathbf{G}_i^T \mathbf{E}_i$ and $\mathbf{G}_i^T \mathbf{G}_i$, we have that

$$\mathcal{N}_\lambda(\rho, \mathbf{P}) \leq \sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(\rho, \mathbf{P}) < 0, \quad \lambda \in \Lambda_N$$

which indicates that a less conservative, and thus more desirable condition would be $\mathcal{N}_\lambda(\rho, \mathbf{P}) < 0$. Unfortunately, a stability condition assured by this inequality does not exist for switched systems in general. For the special case where the matrices $\mathbf{H}_i$, $\mathbf{E}_i$ and $\mathbf{G}_i$ are constant throughout the subsystems, that is, $\mathbf{H}_i = \mathbf{H}$, $\mathbf{E}_i = \mathbf{E}$ and $\mathbf{G}_i = \mathbf{E}$, $\forall i \in \mathbb{K}$, we have $\mathcal{N}_\lambda(\rho, \mathbf{P}) = \sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(\rho, \mathbf{P})$ and a stability condition based on the convex combination of subsystem matrices holds true. Notice that, for the conditions of Theorem 3.5, the same conclusion can be drawn without requiring that matrices $\mathbf{H}_i$ be constant across subsystems.

To implement switching rule (3.65) of Theorem 3.6, the following convex optimization problem

$$\begin{aligned} \min \quad & \rho \\ \text{subject to:} \quad & \mathbf{P} > 0, \ \rho > 0 \\ & \mathbf{Q}_{\lambda_0} \geq 0 \\ & \begin{bmatrix} \mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{Q}_i & \bullet & \bullet \\ \mathbf{H}_i^T \mathbf{P} & -\rho \mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \end{aligned} \tag{3.70}$$

provides the matrices $\mathbf{Q}_i$, for $i \in \mathbb{K}$, as well as matrix $\mathbf{P}$ required.

3.4.2.3 Numerical Example

The following numerical example is supplied to compare the $\mathcal{H}_\infty$ performance index guaranteed by Theorems 3.5 and 3.6, as well as those obtained via numerical simulation.

Example 3.5.

Consider the switched affine system of Example 3.3, with

$$\mathbf{E}_1 = \mathbf{E}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Before implementing the switching rule (3.48) of Theorem 3.5, matrix $\mathbf{P}$ has been calculated by solving the optimization problem (3.52), which in this case yielded

$$\mathbf{P} = \begin{bmatrix} 0.7339 & -0.2187 \\ -0.2187 & 1.2374 \end{bmatrix}$$

along with the guaranteed cost of $J_\infty(\cdot) < 4.0697$. Similarly, for Theorem 3.6, the matrices $\mathbf{P}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ were calculated by solving (3.70), resulting in

$$\mathbf{P} = \begin{bmatrix} 0.7669 & -0.2638 \\ -0.2638 & 1.2982 \end{bmatrix}, \quad \mathbf{Q}_1 = \begin{bmatrix} -13.4815 & 0.5943 \\ 0.5943 & -6.2805 \end{bmatrix}, \quad \mathbf{Q}_2 = \begin{bmatrix} 3.3704 & -0.1486 \\ -0.1486 & 1.5701 \end{bmatrix}$$

associated to the guaranteed cost $J_\infty(\cdot) < 7.1601$, slightly larger than that guaranteed by Theorem 3.5, given the greater conservativeness imposed.

By applying the following disturbance to the system

$$\mathbf{w}(t) = \begin{cases} \sin(\pi t), & 2 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

the trajectories in time for each state can be seen in Figure 3.9, for Theorem 3.5, and Figure 3.10, for Theorem 3.6. Notice that, after the disturbance ceases, the respective switching functions are able to successfully stabilize the systems to the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$. For the $\mathcal{H}_\infty$ case it is not possible to obtain the actual cost $J_\infty(\sigma)$ by

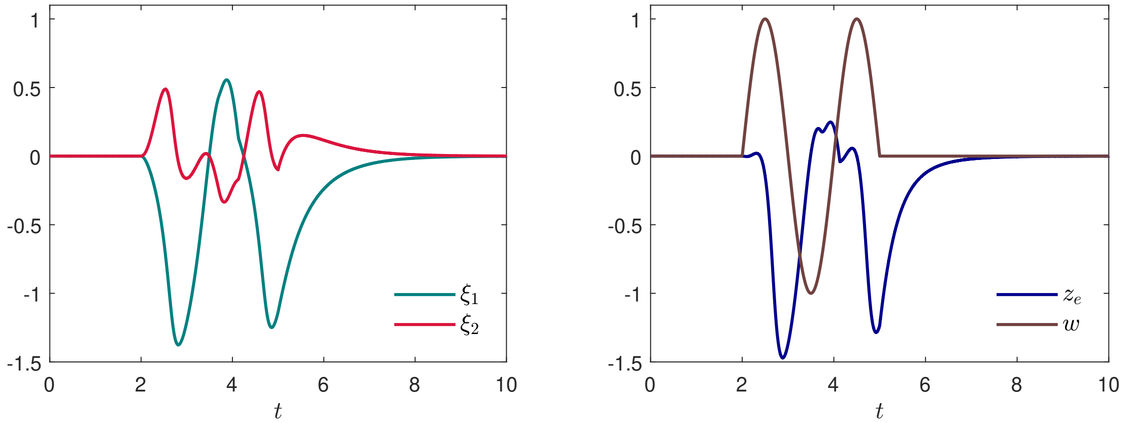


Figure 3.9: Trajectories of each state for the switched affine system under Theorem 3.5.

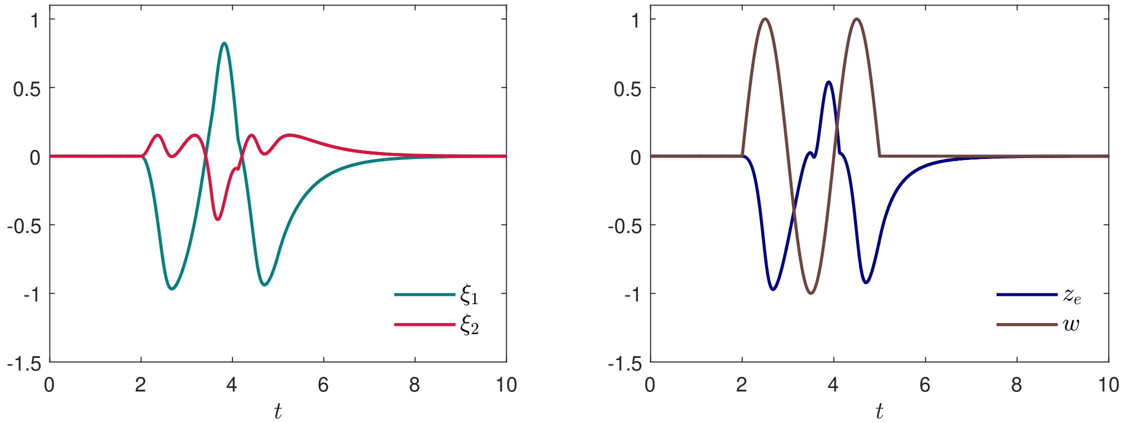


Figure 3.10: Trajectories of each state for the switched affine system under Theorem 3.6.

numerical integration, in contrast with the analysis employed for the $\mathcal{H}_2$ case, since the worst case external input $\mathbf{w}(t)$ for switched system is in general extremely difficult to calculate. ■

Chapter 4

Joint Action Output Feedback Control

The previous chapters introduced underlying concepts that are essential to the main contributions of this work, which consist in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ dynamic output feedback control design of continuous-time switched affine systems by considering the simultaneous design of a set of full order controllers and a switching function. To the best of the author's knowledge, this is the first time that the joint design of two control variables are taken into account in the context of output feedback control of switched affine systems, and have resulted in the following submitted publication [31] in which the contents of this chapter are based upon.

4.1 Introduction

Most results available in the literature treat the control design problem for switched affine systems considering the switching function as the sole control variable to be determined. Some results have been previously mentioned, such as the references [26, 27, 28] which deal with the design of state dependent switching rules, and references [16, 30], which consider an output dependent switching rule. In [30], the switching rule is implemented by means of a full-order switched affine observer, while [16] considers the design of a full-order switched dynamical filter to provide the information needed by the switching function. However fewer references treat the control design problem considering the joint action of a switching rule in tandem with a control law $\mathbf{u}(t)$. The following authors [29] approach this problem by adopting a state dependent switching rule paired with a control law of the form $\mathbf{u}(t) = \mathbf{K}_\sigma \mathbf{x}(t)$, $\sigma \in \mathbb{K}$. This however limits the scope of practical applications in which this technique may be considered for, given that in many situations the state vector is not available for measurement.

Given the above, the control design problem considering two output-dependent control structures is clearly a relevant topic of research, having only been tackled for the case of switched linear systems in [24] and [46]. As such, the results in this chapter generalize the ideas introduced by [16, 24, 29] to deal with the joint design of two stabilizing control variables in the context of switched affine systems, more specifically three techniques based on a full-order switched dynamical controller acting in cooperation with a switching rule, both dependent entirely on the measured output, and guaranteeing global asymptotic stability of the desired equilibrium point are introduced. The first assures an $\mathcal{H}_2$ guaranteed cost while the other two methodologies assure upper bounds for the $\mathcal{H}_\infty$ performance indices with varying degrees of conservativeness.

Some important characteristics of this new methodology are that not only no stability property is expected of individual subsystem matrices $\mathbf{A}_i$, for all $i \in \mathbb{K}$, but it is no longer required that $\mathbf{A}_\lambda \in \mathcal{H}$, thus not restricting the choice of the vector $\lambda \in \Lambda_N$, a requirement which has permeated the results of the previous chapters. This characteristic is very compelling since it allows a larger set of equilibrium points $\mathbf{X}_e$ of the switched system to be attainable, instead of only the subset where $\mathbf{A}_\lambda$ is Hurwitz. This is made possible by joint action of both control structures, and will become clear throughout the course of the chapter.

4.2 Problem Statement

Consider the switched affine system with the following state space representation, already given in its shifted configuration

$$\begin{aligned}\dot{\xi}(t) &= \mathbf{A}_\sigma \xi(t) + \mathbf{B}_\sigma \mathbf{u}(t) + \mathbf{H}_\sigma \mathbf{w}(t) + \boldsymbol{\ell}_\sigma, \quad \xi(0) = 0\\ \mathbf{y}_e(t) &= \mathbf{C}_\sigma \xi(t) + \mathbf{D}_\sigma \mathbf{w}(t)\\ \mathbf{z}_e(t) &= \mathbf{E}_\sigma \xi(t) + \mathbf{F}_\sigma \mathbf{u}(t) + \mathbf{G}_\sigma \mathbf{w}(t)\end{aligned} \tag{4.1}$$

where $\xi(t) \in \mathbb{R}^{n_x}$ is the state vector, considered to be unavailable for measurement, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input to be designed, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the external disturbance, $\mathbf{y}_e(t) \in \mathbb{R}^{n_y}$ is the measured output, $\mathbf{z}_e(t) \in \mathbb{R}^{n_z}$ is the performance output, and $\boldsymbol{\ell}_i = \mathbf{A}_i \mathbf{x}_e + \mathbf{b}_i$ for all $i \in \mathbb{K}$ are the affine terms. Notice that $\mathbf{y}_e(t) = \mathbf{y}(t) - \mathbf{C}_\sigma \mathbf{x}_e$, with $\mathbf{y}(t) = \mathbf{C}_\sigma \mathbf{x}(t) + \mathbf{D}_\sigma \mathbf{w}(t)$, when expressed in terms of $\mathbf{x}(t)$.

Recall from Definition 1 that whenever $\mathbf{b}_i \neq 0$ for some $i \in \mathbb{K}$, the switched system possesses several equilibrium points, characterizing the subset of the state space given by

$$\mathbf{X}_e = \left\{\mathbf{x}_e \in \mathbb{R}^{n_x} : \mathbf{x}_e = -\mathbf{A}_\lambda^{-1} \mathbf{b}_\lambda, \quad \lambda \in \Lambda_N\right\} \tag{4.2}$$

A given choice of $\mathbf{x}_e \in \mathbf{X}_e$, with its associated vector $\lambda_0 \in \Lambda_N$ completes the definition of system (4.1).

Our main objective is to design a control law $\mathbf{u}(t)$, implemented via a full-order switched dynamical controller and dependent of the measured output $\mathbf{y}(t)$, along with a switching function $\sigma(\mathbf{y}_e(t))\colon \mathbb{R}^{n_y} \to \mathbb{K}$ that together are capable of guaranteeing global asymptotic stability of a chosen equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$, or equivalently, $\xi = \mathbf{0}$. These two control structures must also assure upper bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices. To this end, the following full-order switched affine controller is proposed, with state space representation given by

$$\mathcal{C}_\sigma : \begin{cases}\dot{\hat{\xi}}(t) = \hat{\mathbf{A}}_\sigma \hat{\xi}(t) + \hat{\mathbf{B}}_\sigma \mathbf{y}_e(t) + \hat{\boldsymbol{\ell}}_\sigma, \quad \hat{\xi}(0) = 0\\ \mathbf{u}(t) = \hat{\mathbf{C}}_\sigma \hat{\xi}(t)\end{cases} \tag{4.3}$$

where $\hat{\xi} \in \mathbb{R}^{n_x}$ is the state vector of the controller, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control signal, and matrices $\hat{\mathbf{A}}_i$, $\hat{\mathbf{B}}_i$, $\hat{\mathbf{C}}_i$ to be determined, have appropriate dimensions. The controller state $\hat{\xi}$ will not only be used to provide the control signal $\mathbf{u}(t)$, but it will also make possible implementing the switching function $\sigma(\mathbf{y}_e(t))$.

By connecting the controller (4.3) to the switched system (4.1), as illustrated in Figure 4.1, and defining

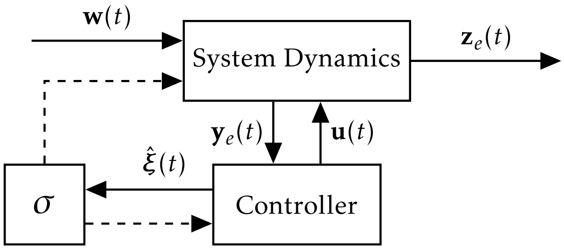


Figure 4.1: Closed-loop system.

the state vector $\tilde{\xi}(t) = \begin{bmatrix} \xi(t)^T & \hat{\xi}(t)^T \end{bmatrix}^T \in \mathbb{R}^{2n_x}$, the following augmented system emerges

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= \tilde{\mathbf{A}}_\sigma \tilde{\xi}(t) + \tilde{\mathbf{H}}_\sigma \mathbf{w}(t) + \tilde{\boldsymbol{\ell}}_\sigma \,, \quad \tilde{\xi}(0) = \mathbf{0} \\ \mathbf{z}_e(t) &= \tilde{\mathbf{E}}_\sigma \tilde{\xi}(t) + \tilde{\mathbf{G}}_\sigma \mathbf{w}(t) \end{aligned} \tag{4.4}$$

with matrices given by

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i\hat{\mathbf{C}}_i \\ \hat{\mathbf{B}}_i\mathbf{C}_i & \hat{\mathbf{A}}_i \end{bmatrix}, \quad \tilde{\boldsymbol{\ell}}_i = \begin{bmatrix} \boldsymbol{\ell}_i \\ \hat{\boldsymbol{\ell}}_i \end{bmatrix}, \quad \tilde{\mathbf{H}}_i = \begin{bmatrix} \mathbf{H}_i \\ \hat{\mathbf{B}}_i\mathbf{D}_i \end{bmatrix} \\ \tilde{\mathbf{E}}_i = \begin{bmatrix} \mathbf{E}_i & \mathbf{F}_i\hat{\mathbf{C}}_i \end{bmatrix}, \quad \tilde{\mathbf{G}}_i = \mathbf{G}_i \tag{4.5}$$

The control design problem consists in determining appropriate conditions that will satisfy Theorems 3.4, 3.5, and 3.6, introduced in Chapter 3, for the augmented system (4.4), thus guaranteeing global asymptotic stability of the closed-loop system, and assuring an upper bound for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices, as defined in (3.35) and (3.44), respectively.

Before proceeding, we first address the problem of obtaining a switching rule that forgoes any dependency on the unknown system state, a result that will be used for the two first theorems, and will be further extended to deal with a switching rule also relying on the disturbance measurement. To accomplish this, we first define block symmetric matrices $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}^{-1}$ as

$$\tilde{\mathbf{P}}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{U} \\ \mathbf{U}^T & \hat{\mathbf{X}} \end{bmatrix}, \quad \tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{Y} & \mathbf{V} \\ \mathbf{V}^T & \hat{\mathbf{Y}} \end{bmatrix} \tag{4.6}$$

such that the relation $\tilde{\mathbf{P}}^{-1}\tilde{\mathbf{P}} = \mathbf{I}$ holds. This implies in the following

$$\mathbf{X}\mathbf{Y} + \mathbf{U}\mathbf{V}^T = \mathbf{I}, \quad \mathbf{X}\mathbf{V} + \mathbf{U}\hat{\mathbf{Y}} = \mathbf{0}, \quad \mathbf{U}^T\mathbf{Y} + \hat{\mathbf{X}}\mathbf{V}^T = \mathbf{0}, \quad \mathbf{U}^T\mathbf{V} + \hat{\mathbf{X}}\hat{\mathbf{Y}} = \mathbf{I} \tag{4.7}$$

and let $\tilde{\mathbf{Q}}_i$ be the block symmetric indefinite matrix

$$\tilde{\mathbf{Q}}_i = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_i \end{bmatrix}, \quad i \in \mathbb{K} \tag{4.8}$$

Now, recall the following switching rule, employed in Theorems 3.4 and 3.6, and applied to the augmented system (4.4)

$$\sigma(\tilde{\xi}) = \arg\min_{i\in\mathbb{K}} -\tilde{\xi}^T\tilde{\mathbf{Q}}_i\tilde{\xi} + 2\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \tag{4.9}$$

Let us first consider the matrix product $\tilde{\xi}^T\tilde{\mathbf{Q}}_i\tilde{\xi}$. Observe that the structure of $\tilde{\mathbf{Q}}_i$ has been defined in order to eliminate any dependency on the unknown system state. Notice that the choice of a constant matrix $\mathbf{M}$ for block $(1,2)$, and a constant symmetric matrix for block $(1,1)$, would also be possible, since although the product $\tilde{\xi}^T\tilde{\mathbf{Q}}_i\tilde{\xi}$ would depend on $\xi(t)$, as such

$$\tilde{\xi}^T\tilde{\mathbf{Q}}_i\tilde{\xi} = \xi^T\mathbf{N}\xi + 2\hat{\xi}^T\mathbf{M}^T\xi + \hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi}, \quad i\in\mathbb{K} \tag{4.10}$$

the dependent terms $\xi^T\mathbf{N}\xi$ and $\hat{\xi}^T\mathbf{M}^T\xi$ are not indexed, and thus, would be constant across $i\in\mathbb{K}$. This would allow for the definition of an equivalent switching function considering only the product $\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi}$. This approach, however, has not shown any advantage with regard to the optimality of the guaranteed cost over the structure of $\tilde{\mathbf{Q}}_i$ originally defined in (4.8). As such, we base this technique on this simpler choice of $\tilde{\mathbf{Q}}_i$.

Now we consider the product $\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i$, as follows

$$\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i = \begin{bmatrix}\xi\\ \hat{\xi}\end{bmatrix}^T \begin{bmatrix}\mathbf{Y} & \mathbf{V}\\ \mathbf{V}^T & \hat{\mathbf{Y}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\ell}_i\\ \hat{\boldsymbol{\ell}}_i\end{bmatrix} \tag{4.11}$$

Notice that by assuring $\mathbf{Y}\boldsymbol{\ell}_i + \mathbf{V}\hat{\boldsymbol{\ell}}_i = \mathbf{0}$, the first n_x rows of term $\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i$ are null, as such, the dependency on the system state is eliminated by making the appropriate choice of $\hat{\boldsymbol{\ell}}_i$ as

$$\hat{\boldsymbol{\ell}}_i = -\mathbf{V}^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \quad \forall i\in\mathbb{K} \tag{4.12}$$

where $\mathbf{V}$ is such that $\exists\mathbf{V}^{-1}$. Observe that this choice also guarantees the nullity of $\tilde{\boldsymbol{\ell}}_{\lambda_0} = \mathbf{0}$, since $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$, as such

$$\tilde{\boldsymbol{\ell}}_{\lambda_0} = \begin{bmatrix}\boldsymbol{\ell}_{\lambda_0}\\ \hat{\boldsymbol{\ell}}_{\lambda_0}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\ell}_{\lambda_0}\\ -\mathbf{V}^{-1}\mathbf{Y}\boldsymbol{\ell}_{\lambda_0}\end{bmatrix} = \mathbf{0} \tag{4.13}$$

The term $\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i$, given in (4.11), with (4.12) and the relations $\hat{\mathbf{Y}} = -\mathbf{U}^{-1}\mathbf{X}\mathbf{V}$ and $\mathbf{V}^T = \mathbf{U}^{-1}(\mathbf{I}-\mathbf{X}\mathbf{Y})$ from (4.7), can be rewritten as

$$\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i = \begin{bmatrix}\xi\\ \hat{\xi}\end{bmatrix}^T \begin{bmatrix}\mathbf{0}\\ \mathbf{V}^T\boldsymbol{\ell}_i + \hat{\mathbf{Y}}\hat{\boldsymbol{\ell}}_i\end{bmatrix} = \hat{\xi}^T(\mathbf{V}^T\boldsymbol{\ell}_i + \hat{\mathbf{Y}}\hat{\boldsymbol{\ell}}_i) = \hat{\xi}^T\mathbf{U}^{-1}\boldsymbol{\ell}_i \tag{4.14}$$

Thus, as desired, the switching rule can be expressed with a dependency solely on the state of the controller $\hat{\xi}$, that is

$$\sigma(\tilde{\xi}) = \sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{U}^{-1}\boldsymbol{\ell}_i \tag{4.15}$$

The problem consists in finding conditions that can be expressed in terms of LMIs, allowing us to obtain the matrices $\hat{\mathbf{A}}_i$, $\hat{\mathbf{B}}_i$, and $\hat{\mathbf{C}}_i$, $i\in\mathbb{K}$, as well as the vectors $\hat{\boldsymbol{\ell}}_i$, needed for implementing the dynamical controller, and matrices $\hat{\mathbf{Q}}_i$ and $\mathbf{U}^{-1}$ important for the switching rule $\sigma(\cdot)$, by solving a convex optimization problem by minimizing the desired performance indices. The following two sections introduce these conditions based on the structures of $\tilde{\mathbf{Q}}_i$ and $\tilde{\boldsymbol{\ell}}_i$ previously defined.

4.3 $\mathcal{H}_2$ Control Design

In this section, we will generalize Theorem 3.4 for the augmented system (4.4) to deal with the two control variables proposed, thus guaranteeing an upper bound for the $\mathcal{H}_2$ performance index. It is assumed that $\mathbf{G}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, in order to work exclusively with strictly proper subsystems. As such, the following theorem can be introduced

Theorem 4.1. *Consider the switched affine system (4.1), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated vector $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}_i$, and $\mathbf{S}$, and matrices $\mathbf{L}_i$ and $\mathbf{W}_i$, for all $i \in \mathbb{K}$, such that*

$$\mathbf{R}_{\lambda_0} \geq 0 \tag{4.16}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & -\mathbf{I} \end{bmatrix} \prec 0, \ \forall i \in \mathbb{K} \tag{4.17}$$

$$\mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \prec 0, \ \forall i \in \mathbb{K} \tag{4.18}$$

$$\begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{H}_j & \mathbf{X} & \bullet \\ \mathbf{Y}\mathbf{H}_j + \mathbf{L}_j\mathbf{D}_j & \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{4.19}$$

with $j = \sigma(0)$, then the following switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i \in \mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i \tag{4.20}$$

along with controller (4.3), whose matrices are given by

$$\begin{aligned} \hat{\mathbf{A}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}(\mathbf{Y}\mathbf{A}_i + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathbf{L}_i\mathbf{C}_i + \mathbf{A}_i^T\mathbf{X}^{-1} + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{E}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1}) \\ \hat{\mathbf{B}}_i &= -(\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{L}_i \\ \hat{\mathbf{C}}_i &= \mathbf{W}_i\mathbf{X}^{-1} \\ \hat{\boldsymbol{\ell}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{Y}\boldsymbol{\ell}_i \\ \hat{\mathbf{Q}}_i &= \mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} \end{aligned} \tag{4.21}$$

make the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and assure the $\mathcal{H}_2$ guaranteed cost

$$J_2(\sigma, \mathcal{C}_\sigma) < \mathrm{tr}(\mathbf{S}) \tag{4.22}$$

for the system.

Proof. The proof consists in demonstrating the validity of Theorem 3.4, whenever the conditions of Theorem 4.1 are satisfied. To this end, consider inequalities (3.39) and (3.40) of Theorem 3.4 expressed in terms of the augmented system (4.4)

$$\tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i + \tilde{\mathbf{Q}}_i \prec 0 \tag{4.23}$$

$$\tilde{\mathbf{Q}}_{\lambda_0} \succeq 0 \tag{4.24}$$

It becomes clear that inequality (4.23) and (4.24) are nonlinear with respect to the matrix variables after substituting the augmented matrices $\tilde{\mathbf{A}}_i$, $\tilde{\mathbf{E}}_i$, $\tilde{\mathbf{P}}$, and $\tilde{\mathbf{Q}}_i$, $i \in \mathbb{K}$. In order to obtain conditions based on LMIs, we introduce the transformation matrix $\tilde{\mathbf{\Gamma}}$ such that

$$\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{U}^T & \mathbf{0} \end{bmatrix} \tag{4.25}$$

First, consider the inequality (4.23) multiplied by the transformation matrix $\tilde{\mathbf{\Gamma}}$ as follows

$$\text{He}\left\{\tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_i\tilde{\mathbf{\Gamma}}\right\} + \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i\tilde{\mathbf{\Gamma}} + \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{Q}}_i\tilde{\mathbf{\Gamma}} \prec 0 \tag{4.26}$$

with intermediary products are given by

$$\tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_i\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{A}_i\mathbf{X} + \mathbf{B}_i\hat{\mathbf{C}}_i\mathbf{U}^T & \mathbf{A}_i \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\hat{\mathbf{C}}_i\mathbf{U}^T + \mathbf{V}\hat{\mathbf{B}}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T & \mathbf{Y}\mathbf{A}_i + \mathbf{V}\hat{\mathbf{B}}_i\mathbf{C}_i \end{bmatrix} \tag{4.27}$$

$$\tilde{\mathbf{E}}_i\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\hat{\mathbf{C}}_i\mathbf{U}^T & \mathbf{E}_i \end{bmatrix} \tag{4.28}$$

$$\tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{Q}}_i\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4.29}$$

Denoting $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{W}_i = \hat{\mathbf{C}}_i\mathbf{U}^T$, inequality (4.26) becomes

$$\begin{bmatrix} \text{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i + (\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) & \bullet \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T + \mathbf{E}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) & \text{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \end{bmatrix} \prec 0 \tag{4.30}$$

Without loss of generality, the choices of $\hat{\mathbf{A}}_i$ in (4.21) and $\mathbf{U} = \mathbf{X}$ can be made. This specific choice of $\hat{\mathbf{A}}_i$ arises from the elimination lemma by making block $(2,1)$ of (4.30) null, and assigning $\mathbf{U} = \mathbf{X}$ introduces no conservativeness, since $\mathbf{V}$ is not a matrix variable in (4.30), as such, from the relations in (4.7), we have that $\mathbf{V} = \mathbf{X}^{-1} - \mathbf{Y}$. From the aforementioned choices, the following inequalities emerge

$$\text{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \prec 0 \tag{4.31}$$

$$\begin{bmatrix} \text{He}\{\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{W}_i\}+\mathbf{R}_i & \bullet \\ \mathbf{E}_i\mathbf{X}+\mathbf{F}_i\mathbf{W}_i & -\mathbf{I} \end{bmatrix} \prec 0 \tag{4.32}$$

Indeed, (4.31) comes directly from block $(2,2)$ of (4.30), and inequality (4.32) is obtained by applying Schur complement on block $(1,1)$ of (4.30) with respect to $(\mathbf{E}_i\mathbf{X}+\mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X}+\mathbf{F}_i\mathbf{W}_i)$. It can be seen that whenever these inequalities are satisfied, and by adopting $\hat{\mathbf{A}}_i$ and $\mathbf{U}$ as above, inequality (4.23) from Theorem 3.4 is verified. Also, observe that (4.16) assures that $\tilde{\mathbf{Q}}_{\lambda_0} \geq 0$, thus satisfying condition (4.24) of Theorem 3.4 for the augmented system (4.4).

Furthermore, notice that the switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i \tag{4.33}$$

comes directly from (4.15) when considering $\mathbf{U}=\mathbf{X}$, as well as the identities (4.21).

Finally, we have that the following inequality

$$\begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{H}_j & \mathbf{X} & \bullet \\ \mathbf{Y}\mathbf{H}_j+\mathbf{L}_j\mathbf{D}_j & \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{4.34}$$

is equivalent to

$$\begin{bmatrix} \mathbf{S} & \bullet \\ \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j & \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} \end{bmatrix} \succ 0 \tag{4.35}$$

with intermediary terms are given by

$$\tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} \tag{4.36}$$

$$\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{H}_j^T & \mathbf{H}_j^T\mathbf{Y}+\mathbf{D}_j^T\mathbf{L}_j^T \end{bmatrix} \tag{4.37}$$

By multiplying inequality (4.35) to the left by $\text{diag}(\mathbf{I},(\tilde{\boldsymbol{\Gamma}}^T)^{-1})$, to the right by its transpose, and subsequently applying Schur complement with respect to $\tilde{\mathbf{P}}$, we obtain

$$\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j \prec \mathbf{S} \tag{4.38}$$

and thus,

$$J_2(\sigma,\mathcal{C}_\sigma) < \text{tr}\left(\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j\right) < \text{tr}(\mathbf{S}) \tag{4.39}$$

with $j=\sigma(0)$, is guaranteed by Theorem 3.4. This concludes the proof. □

This theorem presents a few compelling characteristics. First, notice that the inequalities in (4.17) do not require that the closed-loop matrices $\mathbf{A}_{cl,i}=\mathbf{A}_i+\mathbf{B}_i\mathbf{K}_{c,i}$, with $\mathbf{K}_{c,i}=\mathbf{W}_i\mathbf{X}^{-1}$, be Hurwitz, since matrices $\mathbf{R}_i$ are indefinite. Thus, Theorem 4.1 can assure stability of the switched system even if each individual subsystem is not stabilizable. Also, notice that not only there is no imposition on matrices $\mathbf{A}_i \in \mathcal{H}$, but no convex combination of subsystem matrices $\mathbf{A}_\lambda \in \mathcal{H}$ needs to exit, as was recurrent in the previous chapter. Instead, inequalities (4.16) and (4.17) impose that

$$\sum_{i \in \mathbb{K}} \lambda_{0_i} \left(\mathrm{He}\{\mathbf{A}_{cl,i}\mathbf{X}\} + \mathbf{X}\mathbf{E}_{cl,i}^T \mathbf{E}_{cl,i}\mathbf{X} \right) \prec 0 \tag{4.40}$$

with $\mathbf{E}_{c,i} = \mathbf{E}_i + \mathbf{F}_i \mathbf{K}_{cl,i}$, $i \in \mathbb{K}$. In other words, it is now only necessary that a stable convex combination of the closed loop matrices exists. These considerations bring to light the fact that Theorem 4.1 is able to guarantee global asymptotic stability of the equilibrium point of interest even in the case where a control law $\mathbf{u}(t)$ and a switching rule $\sigma(t)$ are unable to do so when operating independently, allowing for a wider range of problems to be considered. This interesting situation is considered in a numerical example presented towards the end of this chapter. It should be noted that, although the inequalities in (4.18) make necessary that closed loop matrices $\mathbf{A}_i + \mathbf{K}_{o,i}\mathbf{C}_i$, with $\mathbf{K}_{o,i} = \mathbf{Y}^{-1}\mathbf{L}_i$ be quadratically stable with respect to $\mathbf{Y}$, this, however, is not a difficult requirement, since these inequalities are uncoupled from (4.17), through different matrices $\mathbf{Y}$ and $\mathbf{X}$. Furthermore, the matrix gains $\mathbf{K}_{o,i}$ depend in the index i, and thus are independent for each restriction. Finally, notice that, as in the case for Theorem 3.4, an appropriate choice of $\sigma(0)$ can be made so as to optimize the $\mathcal{H}_2$ performance index, or alternatively, by considering $\sigma(0)$ of worst case the $\mathcal{H}_2$ control design problem is made robust with respect to $\sigma(0)$, as previously discussed.

When compared to existing results in the literature, such as [30] and [16], Theorem 4.1 presents more lenient conditions. The first reference requires that all matrices $\mathbf{A}_i$ be Hurwitz, thus being a more conservative result, while the latter requires $\mathbf{A}_\lambda \in \mathcal{H}$, since a control law $\mathbf{u}(t)$ is not taken into account. The following optimization problem, subject to the LMI conditions in Theorem 4.1, provides the matrices necessary to implement the dynamical controller and switching rule, through the relations in (4.21).

$$\begin{aligned}
\min \quad & \mathrm{tr}(\mathbf{S}) \\
\text{subject to:} \quad & \mathbf{R}_{\lambda_0} \succeq 0 \\
& \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \prec 0, \quad \forall i \in \mathbb{K} \\
& \begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \\
& \begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{H}_j & \mathbf{X} & \bullet \\ \mathbf{Y}\mathbf{H}_j + \mathbf{L}_j\mathbf{D}_j & \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0
\end{aligned} \tag{4.41}$$

The next examples aim to illustrate the relevance and usefulness of the proposed methodology. The first reminisces the switched affine system of Example 3.3, and the second is based on the example of [31].

4.3.1 Numerical example: $\mathcal{H}_2$ Control Design

In this example, the switched affine system of Example 3.3 is adopted, with additional inputs $\mathbf{u}(t)$ and $\mathbf{w}(t)$, and outputs $\mathbf{y}_e(t)$ and $\mathbf{z}_e(t)$ as in (4.1).

Example 4.1.

Consider the switched affine system (4.1) with dynamical matrices given in Example 3.3, and with

$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \mathbf{D}_2 = 1, \quad \mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}, \quad \mathbf{G}_1 = \mathbf{G}_2 = \mathbf{0}, \quad \mathbf{F}_1 = \mathbf{F}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

By choosing the equilibrium point $\mathbf{x}_e = [0.2982 \;\; 1.2586]^T$ of interest, associated to $\lambda_0 = [0.4 \;\; 0.6]^T$, it can be noted that the convex combination $\mathbf{A}_{\lambda_0}$ is not Hurwitz, as such, Theorem 3.4 cannot guarantee the global asymptotic stability of the desired equilibrium point, and another approach must be used. A state feedback control design technique may be employed if the state is available for measurement, however in many situations this is not the case. The methodology introduced in this work addresses this scenario, and will be used in this example.

First, the convex optimization problem (4.41) is solved, in order to implement the switching rule (4.20). The following matrices, needed to implement the dynamical controller are obtained via the identities in (4.21).

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} -4.8637 & -4.8656 \\ 1.1615 & 1.1621 \end{bmatrix} \times 10^4, \; \hat{\mathbf{A}}_2 = \begin{bmatrix} -17.7251 & -16.3277 \\ 4.1664 & -7.3999 \end{bmatrix}, \; \hat{\mathbf{B}}_1 = \begin{bmatrix} 4.8646 \\ -1.1617 \end{bmatrix} \times 10^4, \; \hat{\mathbf{B}}_2 = \begin{bmatrix} 11.0817 \\ -0.5102 \end{bmatrix}$$

$$\hat{\mathbf{C}}_1 = \hat{\mathbf{C}}_2 = \begin{bmatrix} -0.2287 & -0.1773 \end{bmatrix}, \; \hat{\boldsymbol{\ell}}_1 = \begin{bmatrix} 1.6111 \\ 12.4843 \end{bmatrix}, \; \hat{\boldsymbol{\ell}}_2 = \begin{bmatrix} -1.0741 \\ -8.3229 \end{bmatrix}$$

as well as the matrices important for the switching rule

$$\hat{\mathbf{Q}}_1 = \begin{bmatrix} -0.2608 & 0.7002 \\ 0.7002 & -1.3657 \end{bmatrix}, \quad \hat{\mathbf{Q}}_2 = \begin{bmatrix} 0.1739 & -0.4668 \\ -0.4668 & 0.9105 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 101.7381 & 10.6627 \\ 10.6627 & 5.5607 \end{bmatrix}$$

along with the guaranteed cost $J_2(\sigma, \mathcal{C}_\sigma) < 34.1051$ for $\sigma(0) = 2$, which minimizes the $\mathcal{H}_2$ index. For the initial condition $\tilde{\xi}_0 = \tilde{\mathbf{H}}_2$, the trajectories in time for each state can be seen in Figure 4.2. The output $\mathbf{z}_e(t)$ and control signal $\mathbf{u}(t)$ are shown in Figure 4.3. Notice that the joint action of both control inputs, $\sigma(\mathbf{y}_e(t))$ and $\mathbf{u}(t)$, were able to asymptotically stabilize the system.

Furthermore, by numerical integration of the product $\mathbf{z}_e(t)'\mathbf{z}_e(t)$, the $\mathcal{H}_2$ cost of the system is calculated as $J_2 = 5.2066 < 34.1051$, within the cost assured by Theorem 4.1, as expected.

Now, only for the sake of illustration, we considered multiple initial conditions distributed around a circle '▬' of radius 10 centered at the origin '×', that is, $\xi_0 = \hat{\xi}_0 = 10 \times [\cos(\theta) \;\; \sin(\theta)]^T$, $\theta \in [0, 2\pi]$ and we have implemented the switching rule and the switched controllers previously determined for the system

$$\dot{\tilde{\xi}} = \tilde{\mathbf{A}}_\sigma \tilde{\xi} + \tilde{\boldsymbol{\ell}}_\sigma, \quad \tilde{\xi}(\mathbf{0}) = \tilde{\xi}_0 \tag{4.42}$$

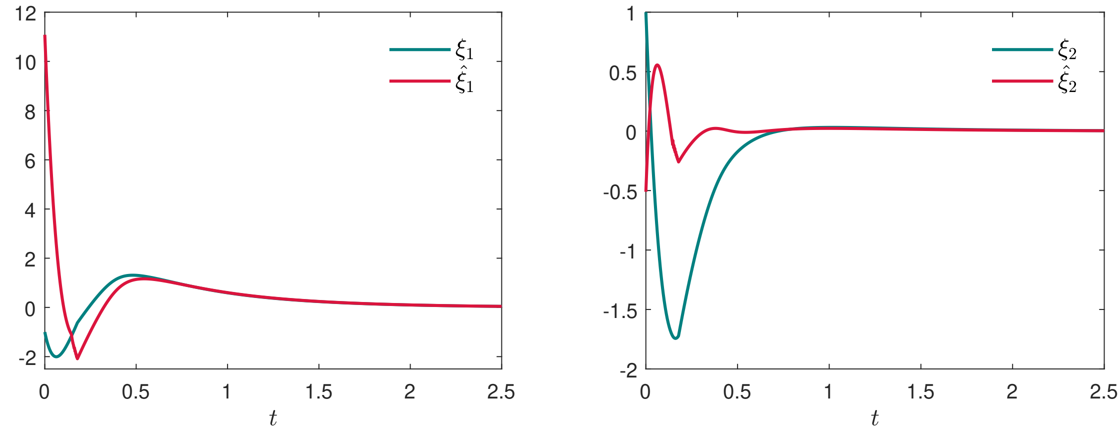


Figure 4.2: Trajectories of each state for the switched affine system under Theorem 4.1.

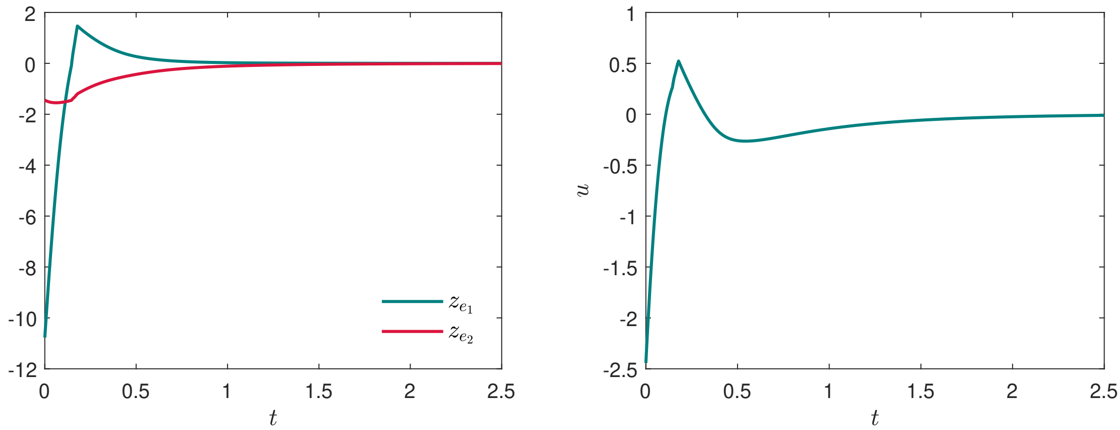
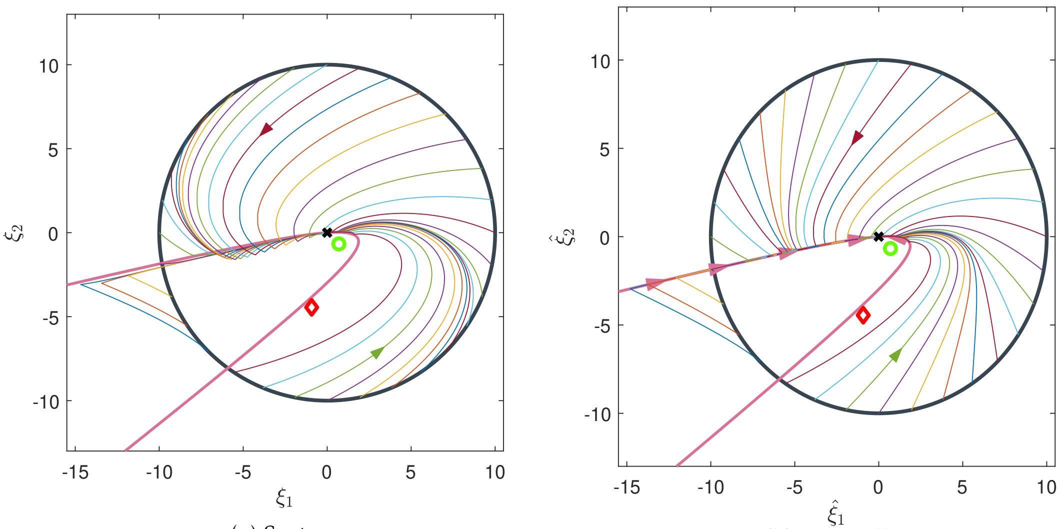


Figure 4.3: Output and control signal for the switched affine system under Theorem 4.1.

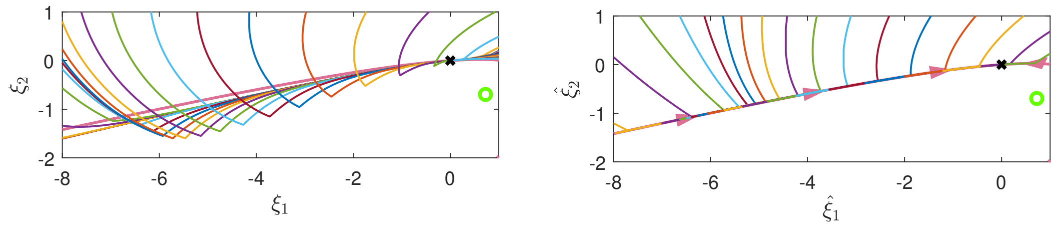
Figure 4.4a shows the phase portrait of the system, and Figure 4.4b for the controller. The equilibrium points of each individual subsystem are indicated by '◇' and '○', respectively. It can be observed that the switching surface, in this case a hyperbole illustrated by the line '—', and given by $-\hat{\xi}^T\hat{\mathbf{Q}}_1\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_1 = -\hat{\xi}^T\hat{\mathbf{Q}}_2\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_2$ only describes the modes of operation of the controller, as it depends solely on the controller state $\hat{\xi}(t)$. In contrast, in Figure 4.4a, it can be seen that the switching events that occur on the system do not coincide with the the switching surface. This behavior is due to the introduced switching rule (4.20), since it relies exclusively on $\hat{\xi}(t)$.

Figures 4.5a and 4.5b show a more detailed phase portrait of the switching surface, showcasing the aforementioned situation. It is interesting to notice that, for some trajectories, when the controller state reaches the switching surface, it evolves via sliding modes in opposite direction to the equilibrium point, until, at a certain instant in time, it begins to asymptotically slide towards the origin, thus stabilizing the overall system at the desired equilibrium point $\mathbf{x}_e$. This behavior, although unexpected, results from the fact that the quadratic Lyapunov function is dependent on the augmented system vector, $v(\tilde{\xi}(t))$, and as a result, its time derivative depends both on the system and controller states. Even in the case of this interesting situation, $\dot{v}(\tilde{\xi}(t)) < 0$ occurs as expected, indicating that the switched system is asymptotically stable. ■



(a) System. (b) Controller.

Figure 4.4: Phase portrait for the switched affine system under Theorem 4.1.



(a) System. (b) Controller.

Figure 4.5: Detailed phase portrait for the switched affine system under Theorem 4.1.

This second example is also available in [31], and illustrates a case in which the proposed technique can guarantee stability of the switched system and an upper bound for the $\mathcal{H}_2$ performance index, where existing results in the literature, for the situation where the system state is unavailable for measurement, cannot.

Example 4.2.

Consider the switched affine system (4.1) comprised of the following three unstable subsystems, for $i \in \{1,2,3\}$

$$\mathbf{A}_1 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -9 \end{bmatrix}, \quad \mathbf{b}_1 = \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{B}_i = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}, \quad \mathbf{H}_i = \mathbf{I}, \quad \mathbf{C}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \quad \mathbf{E}_i = \mathbf{I}, \quad \mathbf{G}_i = \mathbf{0}, \quad \mathbf{D}_i = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{F}_i = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

also available in [31]. It is important to observe that the matrix pairs $(\mathbf{A}_i, \mathbf{B}_i)$ are not controllable for $i \in \{1,2,3\}$, and that there exists no $\lambda \in \Lambda_3$ such that a convex combination $\mathbf{A}_\lambda$ is Hurwitz. As such, only the joint action of a switching function operating alongside a control signal, as proposed in this work, are able to stabilize the system.

Choosing the equilibrium point $\mathbf{x}_e = [0.636 \quad -0.517 \quad 0.237]^T \in \mathbf{X}_e$ of interest, associated to $\lambda_0 = [0.2 \;\; 0.3 \;\; 0.5]^T$, we can proceed with solving the optimization problem (4.41). For the present example, the following matrices were obtained

$$\mathbf{Y} = \begin{bmatrix} 116.8807 & 68.5008 & 110.6348 \\ 68.5008 & 50.1472 & 79.2307 \\ 110.6348 & 79.2307 & 129.5470 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 183.9763 & 126.7359 & 115.2653 \\ 126.7359 & 88.5686 & 80.6768 \\ 115.2653 & 80.6768 & 85.1066 \end{bmatrix}$$

$$\mathbf{W}_1 = \begin{bmatrix} -145.2052 \\ -101.9092 \\ -93.5959 \end{bmatrix}^T, \quad \mathbf{W}_2 = \begin{bmatrix} -145.2252 \\ -101.9232 \\ -93.6103 \end{bmatrix}^T, \quad \mathbf{W}_3 = \begin{bmatrix} -145.1352 \\ -101.8601 \\ -93.5454 \end{bmatrix}^T$$

$$\mathbf{L}_1 = \begin{bmatrix} 0.0956 \\ -1.5061 \\ -1.4760 \end{bmatrix} \times 10^3, \quad \mathbf{L}_2 = \begin{bmatrix} -3.6887 \\ -0.0868 \\ -0.1354 \end{bmatrix} \times 10^4, \quad \mathbf{L}_3 = \begin{bmatrix} -104.7484 \\ -73.1233 \\ -115.8267 \end{bmatrix}$$

used for implementing the dynamic controller and the switching rule considering the relations in (4.21). An associated guaranteed cost of $J_2(\sigma, \mathcal{C}_\sigma) < 18.6502$ for $\sigma(0) = 3$ was also obtained. For the initial condition $\tilde{\xi}_0 = \tilde{\mathbf{H}}_3 \psi_1 = [1 \;\; 0 \;\; 0]^T$, the trajectories in time for each state can be seen in Figure 4.2. The output $\mathbf{z}_e(t)$ and control signal $\mathbf{u}(t)$, are shown in Figure 4.3. The joint action of both control structures were successfully able to

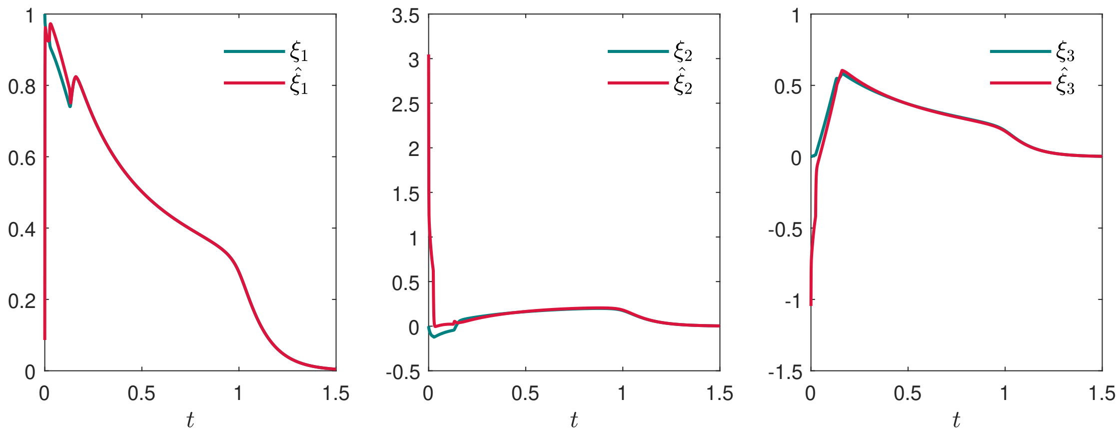


Figure 4.6: Trajectories of each state for the switched affine system under Theorem 4.1 for ψ_1.

stabilize the switched system at the desired equilibrium point $\mathbf{x}_e$, and numerical integration of the product $\mathbf{z}_e(t)'\mathbf{z}_e(t)$ for all three initial conditions $\tilde{\mathbf{H}}_3 \psi_i$, $i \in \{1,2,3\}$, gives the $\mathcal{H}_2$ cost of the system $J_2 = 2.3490 < 18.6502$, lower than the cost assured by Theorem 4.1.

Figure 4.8 shows the switching rule $\sigma(\mathbf{y}_e(t))$ for this example. Notice how at $t \approx 0.13$, the system evolves

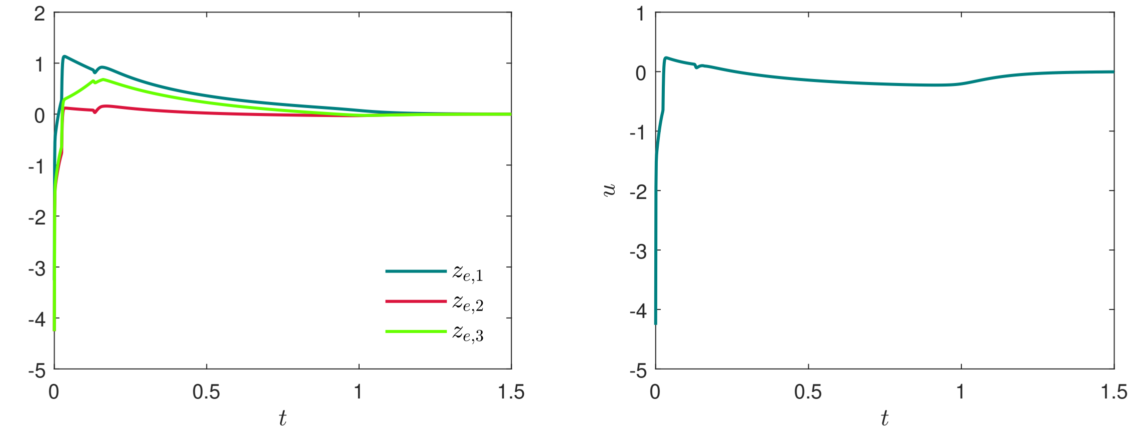


Figure 4.7: Output and control signal for the switched affine system under Theorem 4.1 for ψ_1.

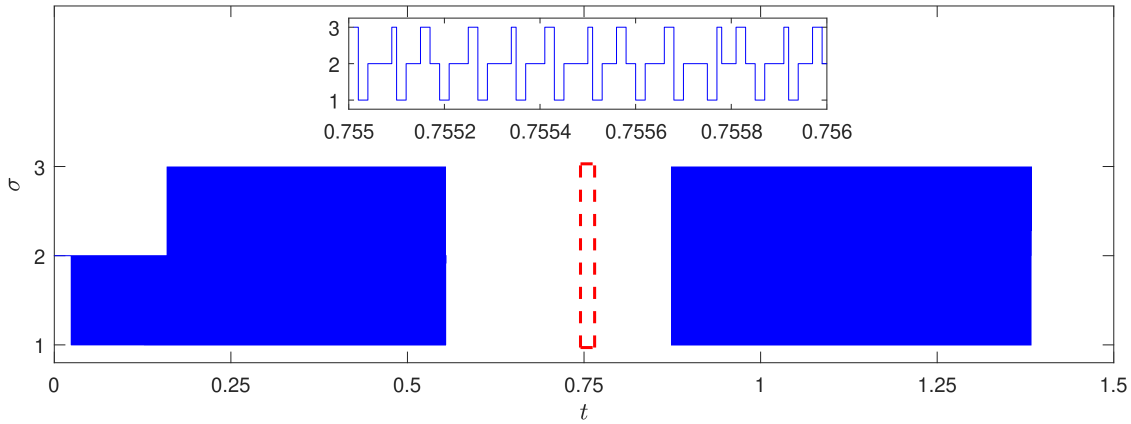


Figure 4.8: Switching rule for the switched affine system under Theorem 4.1 for ψ_1.

towards the equilibrium point in a sliding mode, indicated by the fast switching across the three subsystems. The zoomed-in interval of time in the figure makes clear that the numerical simulation occurs in finite intervals of time, in reality, however, the switching occurs arbitrarily fast. ■

We now present the proposed methodology for the $\mathcal{H}_\infty$ control design problem, based on two different switching functions.

4.4 $\mathcal{H}_\infty$ Control Design

In this section, in order to treat the simultaneous design of the two control structures proposed, namely the switching function and the set of dynamical controllers, that together assure an $\mathcal{H}_\infty$ guaranteed cost, Theorems 3.5 and 3.6 are generalized. Let us recall that for the $\mathcal{H}_\infty$ case, we are concerned with disturbances $\mathbf{w}(t) \in \mathcal{L}_2$. First, the $\mathcal{H}_\infty$ control design problem, considering a switching function that relies solely on output information is introduced. This technique is of much importance, since in many occasions the disturbance is not available for measurement. Nevertheless, in the event that the disturbance is known, a second, less conservative approach is introduced afterwards, which may provide a better $\mathcal{H}_\infty$ performance.

4.4.1 $\mathcal{H}_\infty$ Control Design: Output Dependent Rule

The next theorem generalizes the conditions of Theorem 3.6, and presents the proposed $\mathcal{H}_\infty$ control design technique for the case where the switching rule depends only on the measured output.

Theorem 4.2. *Consider the switched affine system (4.1), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}_i$, matrices $\mathbf{L}_i$ and $\mathbf{W}_i$, for all $i \in \mathbb{K}$, and a scalar ρ, such that*

$$\mathbf{R}_{\lambda_0} \geq 0 \tag{4.43}$$

$$\begin{bmatrix} \mathbf{X} & \bullet \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} > 0 \tag{4.44}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{4.45}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{4.46}$$

then the following switching rule

$$\sigma(\hat{\xi}) = \arg \min_{i \in \mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i \tag{4.47}$$

along with controller (4.3), whose matrices are given by

$$\begin{aligned} \hat{\mathbf{A}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}(\mathbf{Y}\mathbf{A}_i + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathbf{L}_i\mathbf{C}_i + \mathbf{A}_i^T\mathbf{X}^{-1} + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{E}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1} + \boldsymbol{\mathcal{M}}_i(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i)^{-1}\boldsymbol{\mathcal{N}}_i \\ \hat{\mathbf{B}}_i &= -(\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{L}_i \\ \hat{\mathbf{C}}_i &= \mathbf{W}_i\mathbf{X}^{-1} \\ \hat{\boldsymbol{\ell}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{Y}\boldsymbol{\ell}_i \\ \hat{\mathbf{Q}}_i &= \mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} \end{aligned} \tag{4.48}$$

with $\boldsymbol{\mathcal{M}}_i = \mathbf{Y}\mathbf{H}_i + \mathbf{L}_i\mathbf{D}_i + \mathbf{E}_i^T\mathbf{G}_i$ and $\boldsymbol{\mathcal{N}}_i = \mathbf{H}_i^T\mathbf{X}^{-1} + \mathbf{G}_i^T\mathbf{E}_i + \mathbf{G}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1}$, make the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and assure the $\mathcal{H}_\infty$ guaranteed cost

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.49}$$

for the system.

Proof. Similarly to Theorem 4.1, this proof consists in demonstrating the validity of Theorem 3.6, whenever the conditions of Theorem 4.2 are verified. For this, consider inequalities (3.63) and (3.64) of Theorem 3.6, applied to the augmented system (4.4)

$$\tilde{\mathbf{Q}}_{\lambda_0} \succeq 0 \tag{4.50}$$

$$\begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \bullet & \bullet \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} & -\rho\mathbf{I} & \bullet \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.51}$$

In order to linearize inequality (4.51), we again consider the transformation matrix $\tilde{\mathbf{\Gamma}}$ in (4.25), and multiply to the left of (4.51) by $\mathrm{diag}(\tilde{\mathbf{\Gamma}}^T, \mathbf{I}, \mathbf{I})$, and to the right by its transpose, as follows

$$\begin{bmatrix} \tilde{\mathbf{\Gamma}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \tilde{\mathbf{P}}\tilde{\mathbf{H}}_i^T & \tilde{\mathbf{E}}_i^T \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} & -\rho\mathbf{I} & \tilde{\mathbf{G}}_i^T \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Gamma}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.52}$$

whose intermediary products were determined in the proof of Theorem 4.1, and are given in (4.27), (4.28), (4.29), and (4.37). By adopting $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{W}_i = \hat{\mathbf{C}}_i\mathbf{U}^T$, inequality (4.52) can be expressed as

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet & \bullet \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T & \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0 \tag{4.53}$$

Applying Schur complement successively with respect to $-\mathbf{I}$, followed by $(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i)$, the following inequality is obtained

$$\begin{bmatrix} \mathbf{\Xi}_i & \bullet \\ \mathbf{\Omega}_i & \mathbf{\Upsilon}_i \end{bmatrix} \prec 0 \tag{4.54}$$

where

$$\begin{aligned} \mathbf{\Xi}_i = {} & \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i + (\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ & + \left(\mathbf{H}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right)^T \left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i\right)^{-1} \left(\mathbf{H}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right) \end{aligned} \tag{4.55}$$

$$\begin{aligned} \mathbf{\Omega}_i = {} & \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T + \mathbf{E}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ & + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T \left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i\right)^{-1} \left(\mathbf{H}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right) \end{aligned} \tag{4.56}$$

$$\mathbf{\Upsilon}_i = \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T \left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i\right)^{-1} \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right) \tag{4.57}$$

By assigning $\mathbf{U} = \mathbf{X}$ without loss of generality, as previously discussed, and choosing $\hat{\mathbf{A}}_i$ as in (4.48), we have that

$\mathbf{\Omega}_i = \mathbf{0}$, $i \in \mathbb{K}$. Thus, by applying Schur complement appropriately on Ξ_i and Υ_i, we arrive at the inequalities (4.45) and (4.46), respectively. As such, whenever these inequalities are verified, $\Xi_i \prec 0$ and $\Upsilon_i \prec 0$ hold, and (4.53) is satisfied, in turn making condition (4.51) of Theorem 3.6 for the augmented system (4.4) valid.

Also, inequality (4.44) assures $\tilde{\mathbf{P}} \succ 0$, which can be verified by applying the transformation matrix as such $\tilde{\mathbf{\Gamma}}^T \tilde{\mathbf{P}} \tilde{\mathbf{\Gamma}} \succ 0$. In addition, we have that inequality $\mathbf{R}_{\lambda_0} \geq 0$ assures $\tilde{\mathbf{Q}}_{\lambda_0} \geq 0$, verifying condition (4.43) of Theorem 3.6 for the augmented system. Finally, as in Theorem 4.22, the switching rule (4.47) comes from (4.15) when $\mathbf{U} = \mathbf{X}$ is considered, as well as the identities in (4.21). Thus, as guaranteed by Theorem 3.6, the guaranteed cost

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.58}$$

for the augmented system holds, thus concluding the proof. □

As discussed for the $\mathcal{H}_2$ technique, Theorem 4.2 generalizes the results of [16], and eschews the requirement for a stable convex combination of subsystem matrices $\mathbf{A}_\lambda$, as well as the need for the pair $(\mathbf{A}_i, \mathbf{B}_i)$ of the individual subsystems to be controllable. The matrices required to implement the dynamical controller and switching rule proposed in Theorem 4.2, can be calculated numerically via the following convex optimization problem, subject to LMI restrictions

$$\begin{aligned}
\min \quad & \rho \\
\text{subject to:} \quad & \mathbf{R}_{\lambda_0} \geq 0 \\
& \begin{bmatrix} \mathbf{X} & \bullet \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \\
& \begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \\
& \begin{bmatrix} \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K}
\end{aligned} \tag{4.59}$$

It is important to mention that optimization problems subject to LMI restrictions dealing with the minimization of $\mathcal{H}_\infty$ performance may produce ill-conditioned solutions, as discussed in [47]. To overcome this problem, it is proposed that a fixed, suboptimal $\rho > 0$ be supplied, and the objective function $\mathrm{tr}\left(\left(\mathbf{Y} - \mathbf{X}^{-1}\right)^{-1}\right)$ be minimized, as this term multiplies matrices $\hat{\mathbf{A}}_i$, for $i \in \mathbb{K}$. Since the proposed objective function is not linear on the decision variables, consider the LMI restriction

$$\begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{I} & \mathbf{Y} & \bullet \\ \mathbf{0} & \mathbf{I} & \mathbf{X} \end{bmatrix} \succ 0 \tag{4.60}$$

Notice that, by applying Schur complement successively with respect to $\mathbf{X}$, and then $\left(\mathbf{Y} - \mathbf{X}^{-1}\right)$, we obtain

$\left(\mathbf{Y}-\mathbf{X}^{-1}\right)^{-1} \prec \mathbf{S}$. Thus, for a given $\rho > 0$ the following optimization problem

$$\begin{aligned}
\min \quad & \operatorname{tr}(\mathbf{S}) \\
\text{subject to:} \quad & \mathbf{R}_{\lambda_0} \geq 0 \\
& \begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{I} & \mathbf{Y} & \bullet \\ \mathbf{0} & \mathbf{I} & \mathbf{X} \end{bmatrix} \succ 0 \\
& \begin{bmatrix} \operatorname{He}\{\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{W}_i\}+\mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i\mathbf{X}+\mathbf{F}_i\mathbf{W}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \\
& \begin{bmatrix} \operatorname{He}\{\mathbf{Y}\mathbf{A}_i+\mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y}+\mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K}
\end{aligned} \tag{4.61}$$

yields controller matrices with greater numerical stability overall. The next section deals with the $\mathcal{H}_\infty$ control design problem based on less conservative conditions, albeit considering that the disturbance $\mathbf{w}(t)$ is measurable, or known beforehand.

4.4.2 $\mathcal{H}_\infty$ Control Design: Disturbance Measurement

The next theorem generalizes the conditions of Theorem 3.5, and states the proposed $\mathcal{H}_\infty$ control design methodology for the case where the switching rule depends not only on the measured output, but also on the external disturbance.

Theorem 4.3. *Consider the switched affine system (4.1), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}_i$, and $\mathbf{Z}_i$, matrices $\mathbf{J}_i$, $\mathbf{L}_i$ and $\mathbf{W}_i$, for all $i \in \mathbb{K}$, and a scalar $\rho > 0$, such that*

$$\begin{bmatrix} \mathbf{R}_{\lambda_0} & \bullet \\ \mathbf{J}_{\lambda_0}^T & \mathbf{Z}_{\lambda_0} \end{bmatrix} \geq 0 \tag{4.62}$$

$$\begin{bmatrix} \mathbf{X} & \bullet \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{4.63}$$

$$\begin{bmatrix} \operatorname{He}\{\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{W}_i\}+\mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T+\mathbf{J}_i^T & -\rho\mathbf{I}+\mathbf{Z}_i & \bullet \\ \mathbf{E}_i\mathbf{X}+\mathbf{F}_i\mathbf{W}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.64}$$

$$\begin{bmatrix} \operatorname{He}\{\mathbf{Y}\mathbf{A}_i+\mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y}+\mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I}+\mathbf{Z}_i & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.65}$$

then the following switching rule

$$\sigma(\hat{\xi},\mathbf{w}) = \arg\min_{i\in\mathbb{K}} - \begin{bmatrix}\hat{\xi}\\ \mathbf{w}\end{bmatrix}^T \mathcal{Q}_i \begin{bmatrix}\hat{\xi}\\ \mathbf{w}\end{bmatrix} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i \tag{4.66}$$

with

$$\mathcal{Q}_i = \begin{bmatrix}\mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} & \bullet\\ \mathbf{J}_i^T\mathbf{X}^{-1} & \mathbf{Z}_i\end{bmatrix} \tag{4.67}$$

along with controller (4.3), whose matrices are given by

$$\hat{\mathbf{A}}_i = (\mathbf{Y}-\mathbf{X}^{-1})^{-1}(\mathbf{Y}\mathbf{A}_i + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathbf{L}_i\mathbf{C}_i + \mathbf{A}_i^T\mathbf{X}^{-1} + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{E}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1} + \boldsymbol{\mathcal{M}}_i(\rho\mathbf{I}-\mathbf{G}_i^T\mathbf{G}_i-\mathbf{Z}_i)^{-1}\boldsymbol{\mathcal{N}}_i \tag{4.68}$$

$$\hat{\mathbf{B}}_i = -(\mathbf{Y}-\mathbf{X}^{-1})^{-1}\mathbf{L}_i \tag{4.69}$$

$$\hat{\mathbf{C}}_i = \mathbf{W}_i\mathbf{X}^{-1} \tag{4.70}$$

$$\hat{\boldsymbol{\ell}}_i = (\mathbf{Y}-\mathbf{X}^{-1})^{-1}\mathbf{Y}\boldsymbol{\ell}_i \tag{4.71}$$

$$\hat{\mathbf{Q}}_i = \mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} \tag{4.72}$$

with $\boldsymbol{\mathcal{M}}_i = \mathbf{Y}\mathbf{H}_i + \mathbf{L}_i\mathbf{D}_i + \mathbf{E}_i^T\mathbf{G}_i$ *and* $\boldsymbol{\mathcal{N}}_i = \mathbf{H}_i^T\mathbf{X}^{-1} + \mathbf{J}_i^T\mathbf{X}^{-1} + \mathbf{G}_i^T\mathbf{E}_i + \mathbf{G}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1}$, *make the equilibrium point* $\mathbf{x}_e \in \mathbf{X}_e$ *globally asymptotically stable, and assure the* $\mathcal{H}_\infty$ *guaranteed cost*

$$J_\infty(\sigma,\mathcal{C}_\sigma) < \rho \tag{4.73}$$

for the system.

Proof. The proof is based on demonstrating that Theorem 3.5 is valid when the conditions of Theorem 4.3 are satisfied for the augmented system (4.4). But first, we define the following structured matrix

$$\tilde{\mathbf{Q}}_{a,i} = \begin{bmatrix}\tilde{\mathbf{Q}}_i & \tilde{\tilde{\mathbf{Q}}}_i\\ \tilde{\tilde{\mathbf{Q}}}_i^T & \mathbf{Z}_i\end{bmatrix} = \left[\begin{array}{cc|c}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \hat{\mathbf{Q}}_i & \bar{\mathbf{Q}}_i\\ \hline \mathbf{0} & \bar{\mathbf{Q}}_i^T & \mathbf{Z}_i\end{array}\right], \quad \forall i\in\mathbb{K} \tag{4.74}$$

where $\hat{\mathbf{Q}}_i \in \mathbb{R}^{n_x\times n_x}$, $\bar{\mathbf{Q}}_i \in \mathbb{R}^{n_x\times n_w}$, and $\mathbf{Z}_i \in \mathbb{R}^{n_w\times n_w}$. It will become clear that this choice of $\tilde{\mathbf{Q}}_{a,i}$ is important to eliminate the dependency on the unknown system state $\boldsymbol{\xi}(t) \in \mathbb{R}^{n_x}$ by the switching rule (3.48) of Theorem 3.5, expressed in terms of the augmented system, as follows

$$\sigma(\tilde{\xi},\mathbf{w}) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix}\tilde{\xi}\\ \mathbf{w}\end{bmatrix}^T \tilde{\mathcal{L}}_i(\rho,\tilde{\mathbf{P}}) \begin{bmatrix}\tilde{\xi}\\ \mathbf{w}\end{bmatrix} + 2\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \tag{4.75}$$

Observe that condition (3.46) of Theorem 3.5 for the augmented system is equivalent to the inequalities

$$\tilde{\mathcal{L}}_i(\rho,\tilde{\mathbf{P}}) + \tilde{\mathbf{Q}}_{a,i} \prec 0, \quad \forall i\in\mathbb{K} \tag{4.76}$$

together with $\tilde{\mathbf{Q}}_{a,\lambda_0} \geq 0$, where

$$\tilde{\mathcal{L}}_i(\rho,\tilde{\mathbf{P}}) = \begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i & \bullet \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{G}}_i^T\tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i^T\tilde{\mathbf{G}}_i - \rho\mathbf{I} \end{bmatrix}, \quad i \in \mathbb{K} \tag{4.77}$$

By applying Schur complement in (4.76) with respect to $\left(\tilde{\mathbf{G}}_i^T\tilde{\mathbf{G}}_i - \rho\mathbf{I}\right)$, the equivalent inequality

$$\begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \bullet & \bullet \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} + \tilde{\tilde{\mathbf{Q}}}_i^T & -\rho\mathbf{I} + \mathbf{Z}_i & \bullet \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -I \end{bmatrix} < 0 \tag{4.78}$$

is obtained. Multiplying to the left of this inequality by $\mathrm{diag}(\tilde{\boldsymbol{\Gamma}}^T, \mathbf{I}, \mathbf{I})$, and to the right by its transpose, in a similar manner to (4.52), with intermediary products given by (4.27), (4.28), (4.29), and (4.37), and further denoting $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\tilde{\tilde{\mathbf{Q}}}_i^T\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix}\mathbf{J}_i^T & \mathbf{0}\end{bmatrix}$, $\mathbf{J}_i^T = \bar{\mathbf{Q}}_i^T\mathbf{U}^T$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{W}_i = \hat{\mathbf{C}}_i\mathbf{U}^T$ the following inequality emerges

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet & \bullet \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T + \mathbf{J}_i^T & \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} + \mathbf{Z}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} < 0 \tag{4.79}$$

In a similar fashion to Theorem 4.2, we apply Schur complement with respect to $-\mathbf{I}$ followed by $\left(\tilde{\mathbf{G}}_i^T\tilde{\mathbf{G}}_i - \rho\mathbf{I} + \mathbf{Z}_i\right)$, obtaining

$$\begin{bmatrix} \boldsymbol{\Xi}_i & \bullet \\ \boldsymbol{\Omega}_i & \boldsymbol{\Upsilon}_i \end{bmatrix} < 0 \tag{4.80}$$

where

$$\begin{aligned} \boldsymbol{\Xi}_i = {} & \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i + (\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ & + \left(\mathbf{H}_i^T + \mathbf{J}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right)^T\left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i + \mathbf{Z}_i\right)^{-1}\left(\mathbf{H}_i^T + \mathbf{J}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right) \end{aligned} \tag{4.81}$$

$$\begin{aligned} \boldsymbol{\Omega}_i = {} & \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T + \mathbf{E}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ & + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T\left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i + \mathbf{Z}_i\right)^{-1}\left(\mathbf{H}_i^T + \mathbf{J}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right) \end{aligned} \tag{4.82}$$

$$\boldsymbol{\Upsilon}_i = \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T\left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i + \mathbf{Z}_i\right)^{-1}\left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right) \tag{4.83}$$

Choosing $\mathbf{U} = \mathbf{X}$, and $\hat{\mathbf{A}}_i$ as in (4.68), we have $\boldsymbol{\Omega}_i = \mathbf{0}$, $i \in \mathbb{K}$, and, by applying Schur complement as appropriate on $\boldsymbol{\Xi}_i$ and $\boldsymbol{\Upsilon}_i$, the inequalities (4.64) and (4.65) are obtained, respectively. In this manner, whenever these inequalities are satisfied, $\boldsymbol{\Xi}_i < 0$ and $\boldsymbol{\Upsilon}_i < 0$ hold, and (4.79) is verified, thus making condition (4.76) valid. This, along with (4.62) guaranteeing that $\tilde{\mathbf{Q}}_{a,\lambda_0} \geq 0$ satisfies (3.46) of Theorem 3.5 for the augmented system.

Once again, inequality (4.44) assures $\tilde{\mathbf{P}} > 0$. Finally, the switching rule (4.66) with (4.67) comes from (4.75) since

$$\begin{aligned}
\sigma(\tilde{\xi},\mathbf{w}) &= \arg\min_{i\in\mathbb{K}} \begin{bmatrix}\tilde{\xi}\\ \mathbf{w}\end{bmatrix}^T \tilde{\mathcal{L}}_i(\rho,\tilde{\mathbf{P}}) \begin{bmatrix}\tilde{\xi}\\ \mathbf{w}\end{bmatrix} + 2\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \\
&\equiv \arg\min_{i\in\mathbb{K}} - \begin{bmatrix}\tilde{\xi}\\ \mathbf{w}\end{bmatrix}^T \tilde{\mathbf{Q}}_{a,i} \begin{bmatrix}\tilde{\xi}\\ \mathbf{w}\end{bmatrix} + 2\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \\
&= \arg\min_{i\in\mathbb{K}} - \begin{bmatrix}\hat{\xi}\\ \mathbf{w}\end{bmatrix}^T \mathcal{Q}_i \begin{bmatrix}\hat{\xi}\\ \mathbf{w}\end{bmatrix} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i
\end{aligned} \tag{4.84}$$

when considering $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\mathbf{J}_i^T = \bar{\mathbf{Q}}_i^T\mathbf{U}^T$, and $\mathbf{U} = \mathbf{X}$. Also, recalling that the choice $\hat{\boldsymbol{\ell}}_i$ allows for (4.14) which removes any dependency on the system state. In this manner, the guaranteed cost of Theorem 3.5

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.85}$$

is assured for the augmented system. This concludes the proof. □

Theorem 4.3 can be implemented by means of the following convex optimization problem, subject to LMI restrictions.

$$\begin{aligned}
\min \quad & \rho \\
\text{s. to:} \quad & (4.62), (4.63), (4.64), \text{ and } (4.65)
\end{aligned} \tag{4.86}$$

The solution to this problem provides the necessary matrices for the dynamical controller and switching rule, by considering the identities (4.68). Additionally, as previously discussed, to avoid ill-conditioned matrix solutions, by providing a fixed, suboptimal $\rho > 0$, the following optimization problem

$$\begin{aligned}
\min \quad & \mathrm{tr}(\mathbf{S}) \\
\text{s. to:} \quad & \begin{bmatrix}\mathbf{S} & \bullet & \bullet\\ \mathbf{I} & \mathbf{Y} & \bullet\\ \mathbf{0} & \mathbf{I} & \mathbf{X}\end{bmatrix} > 0 \\
& (4.62), (4.64), \text{ and } (4.65)
\end{aligned} \tag{4.87}$$

results in matrices with greater numerical stability for the dynamical controller.

In the next section, we present two examples on $\mathcal{H}_\infty$ Control Design, also based on that of [31], implementing the techniques just introduced.

4.4.3 Numerical example: Output Dependent Rule

This example now illustrates the proposed methodology for the $\mathcal{H}_\infty$ control design problem, also based on the example of [31], considering that the switching rule are dependent only on the measured output.

Example 4.3.

Consider the switched affine system of Example 4.2. In order to implement the switching rule (4.47), and dynamical controller $\mathcal{C}_\sigma$ of Theorem 4.2, matrices $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{W}_i$, $\mathbf{L}_i$, and ρ must be calculated by solving the convex optimization problem (4.86), which, for this example, gives

$$\mathbf{Y} = \begin{bmatrix} 5.8486 & 0.0509 & 1.0488 \\ 0.0509 & 6.5174 & 10.0155 \\ 1.0488 & 10.0155 & 18.2095 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 244.9140 & 169.4628 & 159.3359 \\ 169.4628 & 118.5251 & 111.5770 \\ 159.3359 & 111.5770 & 116.9782 \end{bmatrix}$$

$$\mathbf{W}_1 = \begin{bmatrix} -194.5703 \\ -136.5212 \\ -129.2966 \end{bmatrix}^T, \quad \mathbf{W}_2 = \begin{bmatrix} -194.5709 \\ -136.5216 \\ -129.2970 \end{bmatrix}^T, \quad \mathbf{W}_3 = \begin{bmatrix} -194.5700 \\ -136.5210 \\ -129.2964 \end{bmatrix}^T$$

$$\mathbf{L}_1 = \begin{bmatrix} -2.3160 \\ -61.1186 \\ -65.3486 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} -57.9068 \\ -5.5279 \\ -9.7579 \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} -2.3158 \\ -561.3437 \\ -9.7562 \end{bmatrix}$$

By implementing the controller and rule, via the identities in (4.48), this technique asymptotically stabilizes the switched system to $\boldsymbol{\xi} = \mathbf{0}$, and the guaranteed cost $J_\infty(\sigma, \mathcal{C}_\sigma) < 166.7720$ is assured. Considering the external disturbance $\mathbf{w}(t) = [0 \;\; 5 \;\; 0]^T$ for the interval $0.5 \leq t \leq 1.5$, and zero otherwise, the trajectories in time for each state can be seen in Figure 4.9. The output signal $\mathbf{z}_e(t)$ and control signal $\mathbf{u}(t)$ are shown in Figure 4.10. The

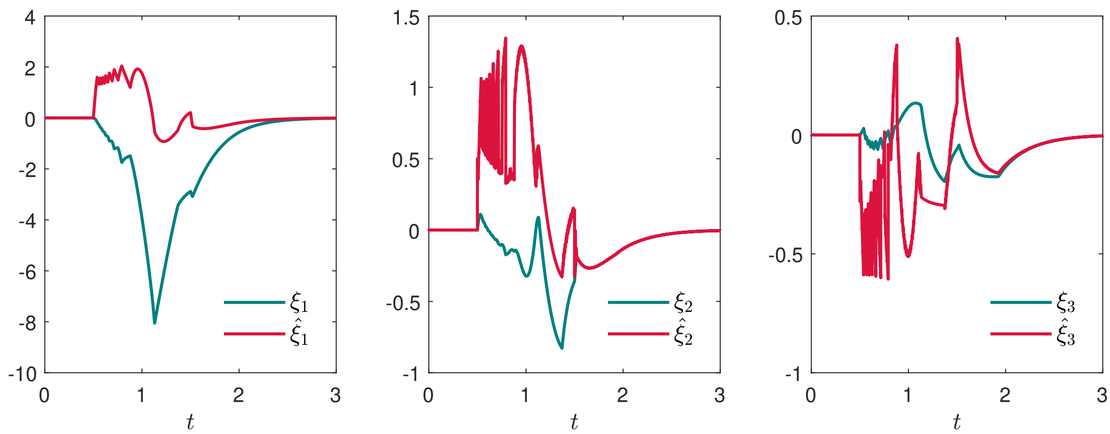


Figure 4.9: Trajectories of each state for the switched affine system under Theorem 4.2.

switching rule of Theorem 4.2 for this example can be seen on Figure 4.11.

The next example will compare these results to that of Theorem 4.3, providing less conservative conditions. ■

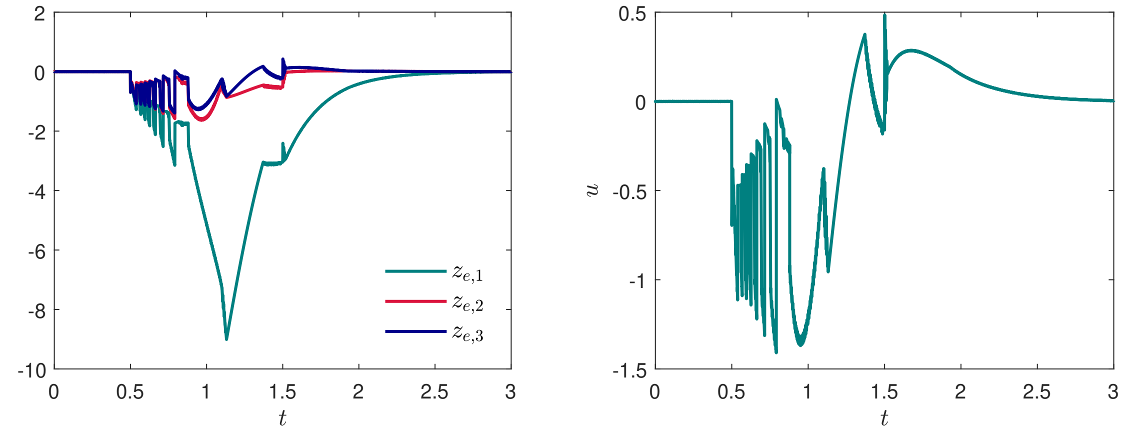


Figure 4.10: Output and control signal for the switched affine system under Theorem 4.2.

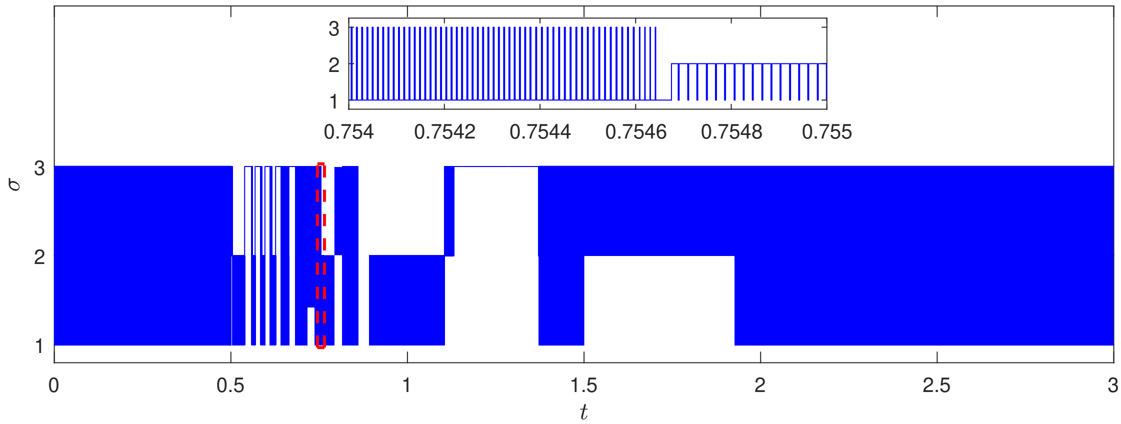


Figure 4.11: Switching rule for the switched affine system under Theorem 4.2.

4.4.4 Numerical example: Disturbance Measurement

This final example considers Example 4.3, now assuming that the disturbance $\mathbf{w}(t)$ is known, and serves as a comparison between the two techniques introduced for the $\mathcal{H}_\infty$ control design problem.

Example 4.4.

Constructing upon Example 4.3, and applying Theorem 4.3, the following matrices are calculated so as to implement the switching rule (4.66) as well as the dynamic controller $\mathcal{C}_\sigma$

$$\mathbf{Y} = \begin{bmatrix} 0.4120 & -0.2157 & 0.6139 \\ -0.2157 & 6.5200 & 10.0130 \\ 0.6139 & 10.0130 & 18.2039 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 246.2547 & 170.3845 & 160.2929 \\ 170.3845 & 119.1525 & 112.2339 \\ 160.2929 & 112.2339 & 117.6591 \end{bmatrix}$$

$$\mathbf{W}_1 = \begin{bmatrix} -195.6495 \\ -137.2608 \\ -130.0659 \end{bmatrix}^T, \quad \mathbf{W}_2 = \begin{bmatrix} -195.6426 \\ -137.2560 \\ -130.0610 \end{bmatrix}^T, \quad \mathbf{W}_3 = \begin{bmatrix} -195.6394 \\ -137.2537 \\ -130.0587 \end{bmatrix}^T$$

$$\mathbf{L}_1 = \begin{bmatrix} -0.2434 \\ -61.3656 \\ -65.5285 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} -5.0956 \\ -1.6023 \\ -2.9202 \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} -0.4222 \\ -11.1887 \\ -17.6459 \end{bmatrix}$$

obtained by solving (4.86) and considering the identities in (4.68). Under Theorem 4.3, the guaranteed cost $J_\infty(\sigma, \mathcal{C}_\sigma) < 38.6818$ is assured. Observe that this guaranteed cost is 76.8% smaller than that assured by Theorem 4.2, due to the reduced conservativeness of its conditions. In this example, we assume that the disturbance is known, thus allowing the switching rule (4.66) to be implemented. Adopting the same external disturbance of Example 4.3, the trajectories in time for each state can be seen in Figure 4.12. The switching rule of Theorem

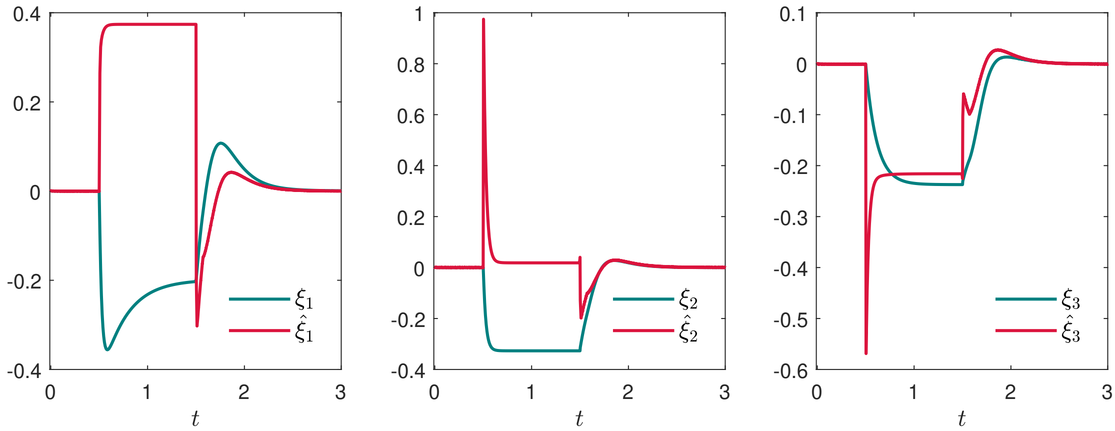


Figure 4.12: Trajectories of each state for the switched affine system under Theorem 4.3.

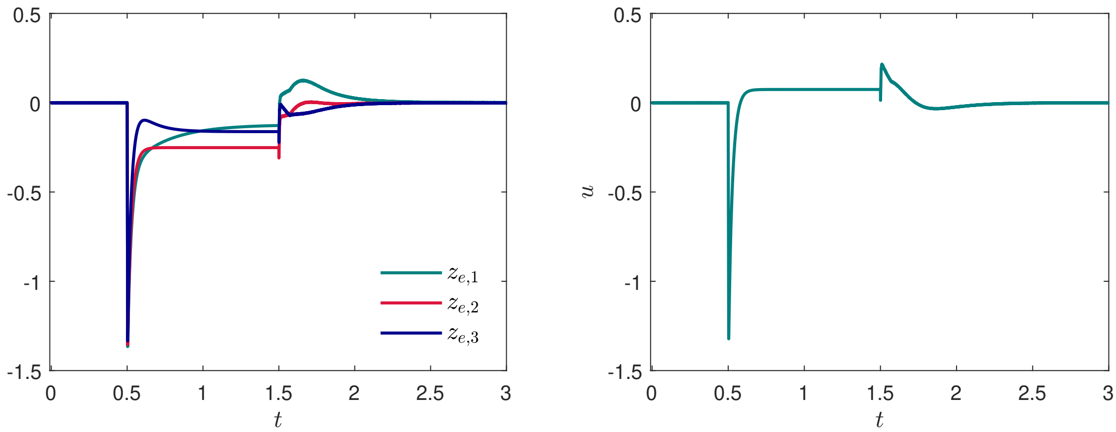


Figure 4.13: Output and control signal for the switched affine system under Theorem 4.3.

4.3 for this example can be seen on Figure 4.14. ∎

Both examples show the effectiveness of the proposed techniques.

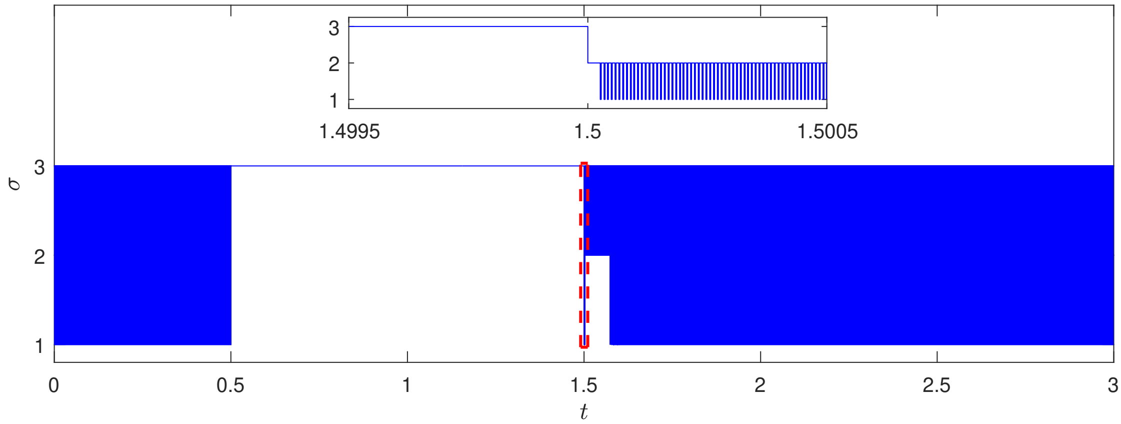


Figure 4.14: Switching rule for the switched affine system under Theorem 4.3.

CHAPTER 5

CONCLUSION

In this work, we have treated the problems of analysis and control design for continuous-time switched affine systems. First, we presented some important concepts with regard to dynamical systems, focusing on their stability properties, as well as on the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for linear time invariant systems. With these concepts on hand, we have introduced the subclass of hybrid systems known as switched systems, and presenting some results already available in the literature on switched linear and affine systems. These results allowed us to understand the particularities and intrinsic characteristics of this important subclass of systems, for instance, the peculiar occurrence of sliding modes, a phenomenon characteristic of switched systems that although sometimes undesirable, due to its very high switching frequency which may lead to equipment damage, it is otherwise crucial when the goal is to attain asymptotic stability of an equilibrium point of interest. In particular, this phenomenon is essential to obtain the asymptotic stability of switched affine systems, since the desired equilibrium point is in general different than those of the individual subsystems, and therefore, an arbitrarily high switching frequency is made necessary to lead and maintain the state trajectories at that point.

For this subclass of hybrid systems in special, the switching function plays a central role, because it can combine the behavior of several different subsystems with the intention of assuring stability and obtaining enhanced performance. On the other hand, the overall dynamical behavior of the system becomes more difficult to analyze mathematically, since it makes the system nonlinear and time-variant. As such, most of the classical tools and methodologies developed for the analysis and control of dynamical systems, some of which were introduced in Chapter 2, cannot be employed anymore. For instance, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms can no longer be used, as the switched system cannot be represented by a transfer function. Given this, novel performances indices have been introduced in the literature, and presented with detail towards the end of Chapter 3.

Despite the many mathematical challenges involved in studying these types of systems, the use of switched systems for practical applications is compelling, since they can model a wide range of real life processes, thus warranting the great interest in this field of research. It is within this scenario that the main contributions of this work are proposed. More specifically, in order to deal with the more realistic case where the state measurements are unavailable, in Chapter 4 we introduce techniques based on the simultaneous design of two control structures, namely a switching function and a set of dynamical controllers, that together guarantee global asymptotic stability as well as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criteria for switched affine systems. The proposed methodology, to the extent of our knowledge, is first being treated in this work. The same control structure has already been adopted in the literature but only for the simpler case of switched linear systems. Compared to the existing approaches on switched affine systems in the literature, the proposed methods allow for a greater number of equilibrium points to be attained by the effect of the switching rule, since the recurrent condition in the literature, requiring that the convex combination of dynamical subsystem matrices $\mathbf{A}_\lambda$ be Hurwitz, is no longer imposed. Instead only the convex combination of closed-loop dynamical matrices must be stable. Furthermore, the proposed techniques are able to guarantee global asymptotic stability of the equilibrium point of interest even in the case where all subsystems are not individually controllable. This reveals the relevance of the joint action of both these control structures for the stability of switched affine systems.

Numerical examples introduced along Chapter 3 illustrate the peculiarities and interesting features of switched systems. Furthermore, examples are provided in Chapter 4 to validate the proposed techniques, and showcase the unique features of our methodologies, and how they contrast with existing results. More specifically, examples considering non-controllable subsystems where no stable convex combination of dynamical matrices exists, are considered. These examples demonstrate that the methods proposed are successful in asymptotically guiding the trajectories of the switched system to the desired equilibrium point, while assuring $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs.

It should be mentioned that the results presented in this work has originated the recent publication [31]. Two further publications, more specifically [48] and [49] introduce techniques for the classical filtering problem for switched affine systems, and will become an integral part of my Master's dissertation. This prominent problem so far has only been treated for the linear case, to the best of our knowledge. The following publications consider either time-dependent switching functions [50, 51, 52], or the joint design of a stabilizing switching function along with a switched filter [53]. However for the more general case of switched affine systems, references [54, 55] only consider the state estimation problem under a switched observer structure. The absence of publications in this topic compelled us to develop a methodology for the classical filter design problem in the context of switched affine systems. The conditions introduced in [48, 49] are based on a full-order switched dynamical filter, which is designed in tandem with an output-dependent stabilizing switching function, collectively assuring an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed cost for the estimation error. Furthermore it is proved that the optimal guaranteed cost filters present an observer-based format, and can be designed independently of the switching function.

Many topics of research in the context of switched affine systems remain to be explored, some of which stand to have much impact for practical applications. Specifically, the development of conditions for a robust control design problem have much relevance for applications in power electronics, as many electrical component values cannot be precisely identified, or depend on operating conditions. Another interesting topic is the development of conditions that guarantee stability considering only a partial information of the equilibrium point of interest assuring global asymptotic stability of the overall system.

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